Math 210 Lecture Notes: Chapter 2.5 2.6 Matrix Product Inverse Matrix

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1. Matrix Product $A \cdot B$

Step 1. The Initial Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}$$

Step 2. Row 1 Column 1

$$\begin{bmatrix} 1 & 2 \\ \ & \ \end{bmatrix} \quad \begin{bmatrix} -1 & \ \\ 2 & \ \end{bmatrix} \quad \begin{bmatrix} ? & \ \\ \ & \ \end{bmatrix}$$

$1 \cdot -1 + 2 \cdot 2 = -1 + 4 = 3$

Step 3. Row 1 Column 2

$$\begin{bmatrix} 1 & 2 \\ \ & \ \end{bmatrix} \quad \begin{bmatrix} \ & 0 \\ \ & 3 \end{bmatrix} \quad \begin{bmatrix} \ & ? \\ \ & \ \end{bmatrix}$$

$1 \cdot 0 + 2 \cdot 3 = 0 + 6 = 6$

Step 4. Row 2 Column 1

$$\begin{bmatrix} \ & \ \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & \ \\ 2 & \ \end{bmatrix} \quad \begin{bmatrix} \ & \ \\ ? & \ \end{bmatrix}$$

$3 \cdot -1 + 4 \cdot 2 = -3 + 8 = 5$

Step 5. Row 2 Column 2

$$\begin{bmatrix} \ & \ \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} \ & 0 \\ \ & 3 \end{bmatrix} \quad \begin{bmatrix} \ & \ \\ \ & ? \end{bmatrix}$$

$3 \cdot 0 + 4 \cdot 3 = 0 + 12 = 12$

Step 6. Matrix Product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 6 \\ 5 & 12 \end{bmatrix}$$

2. Matrix Product $B \cdot A$

Step 1. The Initial Matrices

$$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}$$

Step 2. Row 1 Column 1

$$\begin{bmatrix} -1 & 0 \\ \; & \; \end{bmatrix} \quad \begin{bmatrix} 1 & \; \\ 3 & \; \end{bmatrix} \quad \begin{bmatrix} ? & \; \\ \; & \; \end{bmatrix}$$

$-1 \cdot 1 + 0 \cdot 3 = -1 + 0 = -1$

Step 3. Row 1 Column 2

$$\begin{bmatrix} -1 & 0 \\ \; & \; \end{bmatrix} \quad \begin{bmatrix} \; & 2 \\ \; & 4 \end{bmatrix} \quad \begin{bmatrix} \; & ? \\ \; & \; \end{bmatrix}$$

$-1 \cdot 2 + 0 \cdot 4 = -2 + 0 = -2$

Step 4. Row 2 Column 1

$$\begin{bmatrix} \; & \; \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & \; \\ 3 & \; \end{bmatrix} \quad \begin{bmatrix} \; & \; \\ ? & \; \end{bmatrix}$$

$2 \cdot 1 + 3 \cdot 3 = 2 + 9 = 11$

Step 5. Row 2 Column 2

$$\begin{bmatrix} \; & \; \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} \; & 2 \\ \; & 4 \end{bmatrix} \quad \begin{bmatrix} \; & \; \\ \; & ? \end{bmatrix}$$

$2 \cdot 2 + 3 \cdot 4 = 4 + 12 = 16$

Step 6. Matrix Product

$$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ 11 & 16 \end{bmatrix}$$

3. $A \cdot B$ versus $B \cdot A$

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 12 \end{bmatrix}$

$B \cdot A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 16 \end{bmatrix}$

Conclusion: In general, for matrix multiplication,

$$A \cdot B \neq B \cdot A$$

4. Reparenthesizing

It turns out that

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

provided the dimensions are correct, that is, the number of columns of A = the number of rows of B and the number of columns of B = the number of rows of C.

$\dim(A)$	$\dim(B)$	$\dim(C)$	ABC defined?	$\dim(ABC)$
2×3	3×7	5×7	No	Undefined
5×7	7×2	2×9	Yes	5×9
1×2	2×1	1×3	Yes	1×3

5. Square Matrices

If the number of rows of a matrix A equals the number of columns of A, we call the matrix a square matrix.
That is, a square matrix has dimension $n \times n$.
You can always multiply any square matrix with another (of the same dimension).
Given matrices A, B, C of dimension $n \times n$, we know that all possible products $A \cdot B$, $A \cdot C$, etc. are defined.
You can reparenthesize: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ but, in general, $A \cdot B \neq B \cdot A$

6. Identity Matrices

Then $n \times n$ matrix consisting of 1's down the main diagonal and 0's everywhere else

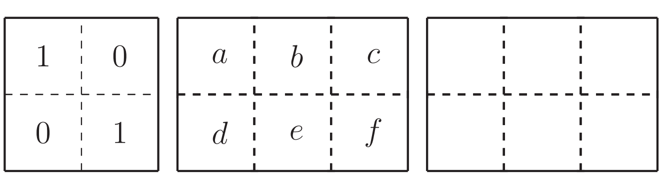
$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

If the value of n is understood, we sometimes omit the subscript and write $I = I_n$ for the identity matrix.
If A is any matrix of dimension $m \times n$, then

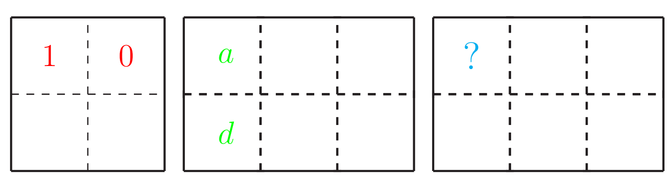
$$I_m \cdot A = A \qquad \text{and} \qquad A \cdot I_n = A.$$

7. Matrix Product $I_2 \cdot A$

Step 1. The Initial Matrices

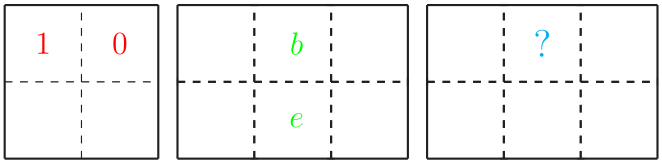


Step 2. Row 1 Column 1



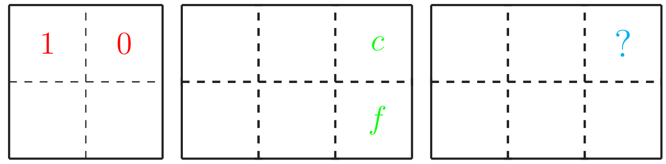
$$1 \cdot a + 0 \cdot d = a + 0 = a$$

Step 3. Row 1 Column 2



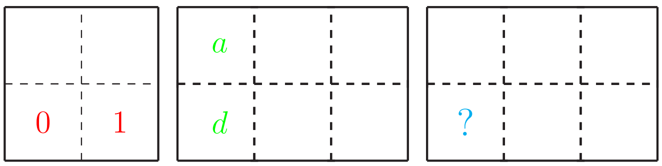
$1 \cdot b + 0 \cdot e = b + 0 = b$

Step 4. Row 1 Column 3



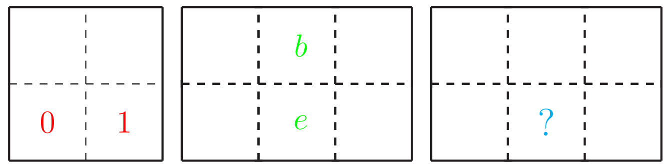
$1 \cdot c + 0 \cdot f = c + 0 = c$

Step 5. Row 2 Column 1



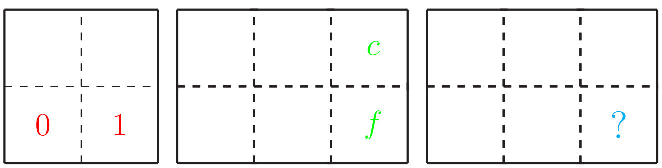
$0 \cdot a + 1 \cdot d = 0 + d = d$

Step 6. Row 2 Column 2



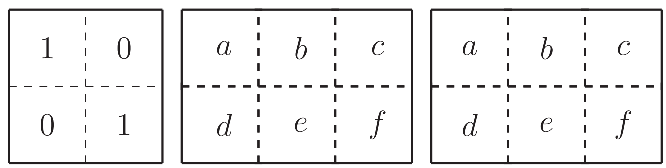
$0 \cdot b + 1 \cdot e = 0 + e = e$

Step 7. Row 2 Column 3



$0 \cdot c + 1 \cdot f = 0 + f = f$

Step 8. Matrix Product



8. Inverse Matrices

Given a square $n \times n$ matrix A. Any matrix B with the property that

$$A \cdot B = B \cdot A = I_n$$

is called the inverse matrix of A.
We write the inverse of A as $B = A^{-1}$.
In general, finding the inverse of a matrix can be difficult.
However, verfiying that a matrix is the inverse is straightforward.

9. Inverse Verification

Show that the inverse matrix of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

10. Inverse Verification

Step 1. The Initial Matrices

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 5 & 3 \\ \hline 1 & 0 & 8 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline -40 & 16 & 9 \\ \hline 13 & -5 & -3 \\ \hline 5 & -2 & -1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

Step 2. Row 1 Column 1

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline -40 & & \\ \hline 13 & & \\ \hline 5 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline ? & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

$1 \cdot -40 + 2 \cdot 13 + 3 \cdot 5 = -40 + 26 + 15 = 1$

Step 3. Row 1 Column 2

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & 16 & \\ \hline & -5 & \\ \hline & -2 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & ? & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

$1 \cdot 16 + 2 \cdot -5 + 3 \cdot -2 = 16 + -10 + -6 = 0$

Step 4. Row 1 Column 3

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & 9 \\ \hline & & -3 \\ \hline & & -1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & ? \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

$1 \cdot 9 + 2 \cdot -3 + 3 \cdot -1 = 9 + -6 + -3 = 0$

Step 5. Row 2 Column 1

$$\begin{array}{|c|c|c|}\hline & & \\ \hline 2 & 5 & 3 \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline -40 & & \\ \hline 13 & & \\ \hline 5 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline ? & & \\ \hline & & \\ \hline \end{array}$$

$2 \cdot -40 + 5 \cdot 13 + 3 \cdot 5 = -80 + 65 + 15 = 0$

Step 6. Row 2 Column 2

$$\begin{array}{|c|c|c|}\hline & & \\ \hline 2 & 5 & 3 \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & 16 & \\ \hline & -5 & \\ \hline & -2 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & ? & \\ \hline & & \\ \hline \end{array}$$

$2 \cdot 16 + 5 \cdot -5 + 3 \cdot -2 = 32 + -25 + -6 = 1$

Step 7. Row 2 Column 3

$$\begin{array}{|c|c|c|}\hline & & \\ \hline 2 & 5 & 3 \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & 9 \\ \hline & & -3 \\ \hline & & -1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & & ? \\ \hline & & \\ \hline \end{array}$$

$2 \cdot 9 + 5 \cdot -3 + 3 \cdot -1 = 18 + -15 + -3 = 0$

Step 8. Row 3 Column 1

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline 1 & 0 & 8 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline -40 & & \\ \hline 13 & & \\ \hline 5 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline ? & & \\ \hline \end{array}$$

$1 \cdot -40 + 0 \cdot 13 + 8 \cdot 5 = -40 + 0 + 40 = 0$

Step 9. Row 3 Column 2

1	0	8

	16	
	−5	
	−2	

	?	

$1 \cdot 16 + 0 \cdot -5 + 8 \cdot -2 = 16 + 0 + -16 = 0$

Step 10. Row 3 Column 3

1	0	8

		9
		−3
		−1

		?

$1 \cdot 9 + 0 \cdot -3 + 8 \cdot -1 = 9 + 0 + -8 = 1$

Step 11. Matrix Product

1	2	3
2	5	3
1	0	8

−40	16	9
13	−5	−3
5	−2	−1

1	0	0
0	1	0
0	0	1

11. Solving Equations

Consider the system of equations:
$x + 2y + 3z = 1$
$2x + 5y + 3z = 2$
$x + 8z = -2$
To write this as a single matrix equation we need
• the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

the column matrices of • variables and • constants:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

12. Solving Equations

The matrix equation $AX = B$ now becomes

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Computing the product of the 3×3 matrix A with the 3×1 column matrix X gives

$$\begin{bmatrix} x + 2y + 3z \\ 2x + 5y + 3z \\ x + 8z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Since two matrices are equal if and only if their entries are identical, we arrive at the original three equations:
$x + 2y + 3z = 1$, $2x + 5y + 3z = 2$, $x + 8z = -2$

13. What's the point?

The point is that the system of three equations

$$x + 2y + 3z = 1$$
$$2x + 5y + 3z = 2$$
$$x + 8z = -2$$

is equivalent to the single matrix equation

$$AX = B$$

It looks like we have just rephrased the original problem.
How does this help us find the actual solution?

14. Solving the matrix equation

The key is the inverse matrix.
We can solve the matrix equation $AX = B$ if we know the inverse matrix A^{-1}:

$AX = B$	original matrix equation
$A^{-1}(AX) = A^{-1}B$	multiply by A^{-1}
$(A^{-1}A)X = A^{-1}B$	reparenthesize
$I_3X = A^{-1}B$	definition of inverse
$X = A^{-1}B$	property of identity matrix

15. A Comparison

It is interesting to compare the matrix method side by side with the standard way of solving a linear equation $ax = b$ in algebra.

Matrix Equation	Ordinary Algebra Equation
$AX = B$	$ax = b$
$A^{-1}(AX) = A^{-1}B$	$\frac{1}{a}(ax) = \frac{1}{a}b$
$(A^{-1}A)X = A^{-1}B$	$(\frac{1}{a} \cdot a)x = \frac{1}{a}b$
$I_3X = A^{-1}B$	$1 \cdot x = \frac{1}{a}b$
$X = A^{-1}B$	$x = \frac{b}{a}$

16. Grand Finale

Using the value of A^{-1} which we verified earler, the solution to our matrix equation is

$$X = A^{-1}B = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-40)(1) + (16)(2) + (9)(-2) \\ (13)(1) + (-5)(2) + (-3)(-2) \\ (5)(1) + (-2)(2) + (-1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -40 + 32 - 18 \\ 13 - 10 + 6 \\ 5 - 4 + 2 \end{bmatrix} = \begin{bmatrix} -26 \\ 9 \\ 3 \end{bmatrix}$$

17. Advantages

How is this method superior to just row reducing the matrix of the system?
Consider the system

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 3z = 2 \\ x + 8z = -2 \end{cases}$$

In the row reduction method, if the constants 1, 2, and -2 on the right of the equal sign, then the new system matrix must be row reduced.
But with the inverse matrix method, we just need to make a second simple matrix multiplication A^{-1} times the new column matrix of constants.

18. Finding A^{-1}

Okay, I'm sold. But how do I go about finding the inverse matrix? After all, I can't expect somebody to tell me what it is every time!
Unfortunately, finding A^{-1} is hard. How hard exactly, depends on the dimension of A.
For a 2×2 matrix, there is a formula for the inverse.
For 3×3 matrices, or larger, there is a procedure using row reduction for finding the inverse.

19. The 2 × 2 Case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that

- Entries a and d are switched
- Entries b and c are negative
- The matrix is multiplied by $\frac{1}{ad-bc}$
- If $ad-bc=0$, then the inverse does not exist.

21. The 3 × 3 Case

To find the inverse of a 3 by 3 matrix A, row reduce the 3 by 6 matrix

$$[A \mid I_3]$$

The left half of this matrix is A, the matrix whose inverse we want to find.
The right half of this matrix is just the 3 by 3 identity matrix I_3.
After row reduction, the reduced matrix should look like

$$[I_3 \mid B]$$

The matrix B is the inverse of A. Pure Magic.

20. A 2 × 2 Example

By the formula,

$$\begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -7 \\ -1 & 2 \end{bmatrix}$$

To Verify:

$$\frac{1}{3}\begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & -7 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2\cdot 5 - 7\cdot 1 & 2(-7)+7\cdot 2 \\ 1\cdot 5 + 5(-1) & 1(-7)+5\cdot 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I_2$$

22. The 3 × 3 Case

Step 1. The Initial Matrix.

1	2	3	1	0	0
2	5	3	0	1	0
1	0	8	0	0	1

Step 2.

1	2	3	1	0	0
2	5	3	0	1	0
1	0	8	0	0	1

Pivot at row 1 col 1

Step 3

1	2	3	1	0	0
2	5	3	0	1	0
1	0	8	0	0	1

Step 4

1	2	3	1	0	0
0	1	-3	-2	1	0
1	0	8	0	0	1

Add $-2 \times$ row 1 to row 2

$$-2 \times 1 + 2 = 0$$

$$-2 \times 2 + 5 = 1$$

$$-2 \times 3 + 3 = -3$$

$$-2 \times 1 + 0 = -2$$

Step 5

1	2	3	1	0	0
0	1	-3	-2	1	0
1	0	8	0	0	1

Step 6

1	2	3	1	0	0
0	1	-3	-2	1	0
0	-2	5	-1	0	1

Add $-1 \times$ row 1 to row 3

$$-1 \times 1 + 1 = 0$$

$$-1 \times 2 + 0 = -2$$

$$-1 \times 3 + 8 = 5$$

$$-1 \times 1 + 0 = -1$$

Step 7

1	2	3	1	0	0
0	1	-3	-2	1	0
0	-2	5	-1	0	1

Pivot at row 2 col 2

Step 8

1	2	3	1	0	0
0	1	-3	-2	1	0
0	-2	5	-1	0	1

Add $-2 \times$ row 2 to row 1

$$-2 \times 1 + 2 = 0$$

$$-2 \times -3 + 3 = 9$$

$$-2 \times -2 + 1 = 5$$

$$-2 \times 1 + 0 = -2$$

Step 9

1	0	9	5	-2	0
0	1	-3	-2	1	0
0	-2	5	-1	0	1

Step 10

1	0	9	5	-2	0
0	1	-3	-2	1	0
0	-2	5	-1	0	1

Add $2 \times$ row 2 to row 3

$$2 \times 1 + -2 = 0$$

$$2 \times -3 + 5 = -1$$

$$2 \times -2 + -1 = -5$$

$$2 \times 1 + 0 = 2$$

Step 11.

1	0	9	5	−2	0
0	1	−3	−2	1	0
0	0	−1	−5	2	1

Step 12.

1	0	9	5	−2	0
0	1	−3	−2	1	0
0	0	−1	−5	2	1

Pivot at row 3 col 3

Step 13.

1	0	9	5	−2	0
0	1	−3	−2	1	0
0	0	−1	−5	2	1

Multiply row 3 by -1

Step 14.

1	0	9	5	−2	0
0	1	−3	−2	1	0
0	0	1	5	−2	−1

Step 15.

1	0	9	5	−2	0
0	1	−3	−2	1	0
0	0	1	5	−2	−1

Add $-9 \times$ row 3 to row 1

$$-9 \times 1 + 9 = 0$$

$$-9 \times 5 + 5 = -40$$

$$-9 \times -2 + -2 = 16$$

$$-9 \times -1 + 0 = 9$$

Step 16.

1	0	0	−40	16	9
0	1	−3	−2	1	0
0	0	1	5	−2	−1

Step 17.

1	0	0	-40	16	9
0	1	-3	-2	1	0
0	0	1	5	-2	-1

Add $3 \times$ row 3 to row 2

$$3 \times 1 + -3 = 0$$

$$3 \times 5 + -2 = 13$$

$$3 \times -2 + 1 = -5$$

$$3 \times -1 + 0 = -3$$

Step 18.

1	0	0	-40	16	9
0	1	0	13	-5	-3
0	0	1	5	-2	-1