Math 210 Lecture Notes: Chapter 2.2 Solving Systems by Matrices

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1. Solving Systems by Matrices

It is clear from the example that

- We need to work systematically.
- The variables (x, y, z) are just placeholders.
- We only need the values of the coefficients of the variables.
- Our "method" should be implementable on a computer–
- especially when the number of variables is > 3.
- Our method uses a rectangular array of numbers, called a matrix (plural "matrices").
- Each row of the matrix contains all the information about a single equation in our system.

2. Translating equations

First, write each equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

as a row

$$a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n \quad b$$

in the matrix.
Be careful to write the coefficients in order.
If there is a missing term in the equation, the zero coefficient must be present in the matrix.
If the coefficient is one, it is usually not written.
For example, $x+2y-z$ is really $1x+2y+(-1)z$.

3. Example

If there are four variables x_1, x_2, x_3, x_4, the equation

$$8 - 5x_4 + x_2 - x_1 = 10$$

should be rewritten

$$-x_1 + x_2 - 5x_4 = 2$$

Putting in the missing coefficients of 0 and 1:

$$-1x_1 + 1x_2 + 0x_3 - 5x_4 = 2$$

So the row corresponding to this equation is

$$-1 \quad 1 \quad 0 \quad -5 \quad 2$$

4. Translating a System

Write a row for each equation in your system.
For example, the system

$$\begin{aligned} 3x_1 + 6x_2 - 5x_3 &= 0 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 9 \end{aligned}$$

is written as the matrix

$$\begin{bmatrix} 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \end{bmatrix}$$

Later we will see how to solve this system just by using the matrix.

5. Row Operations

- switch any two rows
- multiply any row by a non-zero constant
- add any multiple of one row to another

6. Observations

- By multiple switches you can re-arrange the rows in any order you want.
- You cannot multiply a row by 0, otherwise you will lose all the numerical information contained in that row.
- You can add 0 times one row to another.

  But it would be a waste of time.
  The matrix would be unchanged.

7. Why these three Rules

These important fact about these three rules is that the corresponding system of equations obtained by applying each of these rules to a matrix is equivalent to original system, meaning the two systems have exactly the same solutions.
Consider the first rule: you may switch any two rows. Clearly switching any two equations in your system does not alter the solution.

8. Example

$$\begin{aligned} 3x_1 + 6x_2 - 5x_3 &= 0 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 9 \end{aligned}$$

and

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

have the same set of solutions.

9. First Steps

When we work systematically to solve a system, our first goal is to eliminate the first variable x_1 from all but the first equation.
That is, we want the coefficient of x_1 in the first equation to be 1 and the coefficient of x_1 in the second, third, etc, equations to be zero.
Since the coefficents of x_1 are all in the first column of the matrix, we concentrate on column 1 (ignoring the other columns for now)
In terms of the matrix of the system, we want the first column to look like: $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

10. Pivoting

We can turn the first coefficient a_1 of the first equation into a 1 by multiplying the entire equation by $\dfrac{1}{a_1}$ provided a_1 is not zero.
If $a_1 = 0$ then we must switch row 1 (or equation 1) with a row which has a nonzero entry in column 1 (corresponding to an equation whose x_1 coefficient is nonzero).
Once we have a one in position (row 1, col 1) we can add multiples of that row to create zeros in the first column for all other entries below row 1.
This process is called pivoting (at position (1,1)).

11. Two Minor Points

Point 1. If the coefficient of x_1 is zero in all the equations, then there is not much we can do. We need to abandon x_1 and consider the second variable x_2.

Point 2. We may consider switching rows if another row has a 1 in the first column. This can eliminate (or postpone) working with fractions.

Example. In the system with matrix $\begin{bmatrix} 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \end{bmatrix}$ we could multiply row 1 by $\frac{1}{3}$ to create a 1 in row 1, col 1: $\begin{bmatrix} \boxed{1} & 2 & -\frac{5}{3} & 0 \\ 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \end{bmatrix}$ Notice the fraction $-\frac{5}{3}$.

12. Avoiding Fractions

But since the third row (equation) of the matrix $\begin{bmatrix} 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \end{bmatrix}$ has a leading 1, it is simpler to switch row 1 and row 3: $\begin{bmatrix} \boxed{1} & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$ and avoid dealing with fractions—at least for now.

13. Completing the pivot

We wish to pivot at entry in position (row 1, col 1) $\begin{bmatrix} \boxed{1} & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$

To make a zero in position (row 2, col 1) we add -2 times row 1 to row 2: $\begin{bmatrix} 1 & 1 & 2 & 9 \\ \boxed{2} & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$

To make a zero in position (row 3, col 1) we add -3 times row 1 to row 3: $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ \boxed{3} & 6 & -5 & 0 \end{bmatrix}$

14. Solving the system

Once we have completed the pivot at entry (1,1), we move on to the next column (or variable).

By switching rows, if necessary, we next pivot in position (2,2). Our final pivot is in position (3,3), resulting in a matrix which looks like: $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$

The system of equations represented by this final solution is:

$$1x_1 + 0x_2 + 0x_3 = a$$

$$0x_1 + 1x_2 + 0x_3 = b$$

$$0x_1 + 0x_2 + 1x_3 = c$$

giving us the solution: $x_1 = a$, $x_2 = b$, $x_3 = c$.

15. Example 1

Step 1. The Initial Matrix.

3	6	−5	0
2	4	−3	1
1	1	2	9

Step 2.

3	6	−5	0
2	4	−3	1
1	1	2	9

Switch rows 1 and 3

Step 3

1	1	2	9
2	4	−3	1
3	6	−5	0

Step 4

1	1	2	9
2	4	−3	1
3	6	−5	0

Pivot at row 1 col 1

Step 5

1	1	2	9
2	4	−3	1
3	6	−5	0

Add $-2 \times$ row 1 to row 2

$$-2 \times 1 + 2 = 0$$

$$-2 \times 1 + 4 = 2$$

$$-2 \times 2 + -3 = -7$$

$$-2 \times 9 + 1 = -17$$

Step 6

1	1	2	9
0	2	−7	−17
3	6	−5	0

Step 7

1	1	2	9
0	2	−7	−17
3	6	−5	0

Add −3 × row 1 to row 3

$-3 \times 1 + 3 = 0$

$-3 \times 1 + 6 = 3$

$-3 \times 2 + -5 = -11$

$-3 \times 9 + 0 = -27$

Step 8

1	1	2	9
0	2	−7	−17
0	3	−11	−27

Step 9

1	1	2	9
0	2	−7	−17
0	3	−11	−27

Pivot at row 2 col 2

Step 10

1	1	2	9
0	2	−7	−17
0	3	−11	−27

Multiply row 2 by $\frac{1}{2}$

Step 11.

1	1	2	9
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	3	-11	-27

Step 12.

1	1	2	9
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	3	-11	-27

Add $-1 \times$ row 2 to row 1

$$-1 \times 1 + 1 = 0$$

$$-1 \times -\frac{7}{2} + 2 = \frac{11}{2}$$

$$-1 \times -\frac{17}{2} + 9 = \frac{35}{2}$$

Step 13.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	3	-11	-27

Step 14.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	3	-11	-27

Add $-3 \times$ row 2 to row 3

$$-3 \times 1 + 3 = 0$$

$$-3 \times -\frac{7}{2} + -11 = -\frac{1}{2}$$

$$-3 \times -\frac{17}{2} + -27 = -\frac{3}{2}$$

Step 15.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$

Step 16.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$

Pivot at row 3 col 3

Step 17.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$

Multiply row 3 by -2

Step 18.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	1	3

Step 19.

1	0	$\frac{11}{2}$	$\frac{35}{2}$
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	1	3

Add $-\frac{11}{2}$ × row 3 to row 1

$-\frac{11}{2} \times 1 + \frac{11}{2} = 0$

$-\frac{11}{2} \times 3 + \frac{35}{2} = 1$

Step 20.

1	0	0	1
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	1	3

Step 21.

1	0	0	1
0	1	$-\frac{7}{2}$	$-\frac{17}{2}$
0	0	1	3

Add $\frac{7}{2}$ × row 3 to row 2

$\frac{7}{2} \times 1 + -\frac{7}{2} = 0$

$\frac{7}{2} \times 3 + -\frac{17}{2} = 2$

Step 22. The Reduced Matrix

1	0	0	1
0	1	0	2
0	0	1	3

Solution: $x = 1$, $y = 2$, $z = 3$

16. Example 2

Step 1. The Initial Matrix.

3	-1	-1	1
7	1	-1	6
2	1	-1	2

Step 2.

$\boxed{3}$	-1	-1	1
7	1	-1	6
2	1	-1	2

Pivot at row 1 col 1

Step 3.

3	-1	-1	1
7	1	-1	6
2	1	-1	2

Step 4.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
7	1	-1	6
2	1	-1	2

Multiply row 1 by $\frac{1}{3}$

Step 5.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
7	1	-1	6
2	1	-1	2

Step 6.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{11}{3}$
2	1	-1	2

Add $-7 \times$ row 1 to row 2

$$-7 \times 1 + 7 = 0$$

$$-7 \times -\frac{1}{3} + 1 = \frac{10}{3}$$

$$-7 \times -\frac{1}{3} + -1 = \frac{4}{3}$$

$$-7 \times \frac{1}{3} + 6 = \frac{11}{3}$$

Step 7.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{11}{3}$
2	1	-1	2

Add $-2 \times$ row 1 to row 3

$$-2 \times 1 + 2 = 0$$

$$-2 \times -\frac{1}{3} + 1 = \frac{5}{3}$$

$$-2 \times -\frac{1}{3} + -1 = -\frac{1}{3}$$

$$-2 \times \frac{1}{3} + 2 = \frac{4}{3}$$

Step 8.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Step 9.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	$\boxed{\frac{10}{3}}$	$\frac{4}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Pivot at row 2 col 2

Step 10.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Multiply row 2 by $\frac{3}{10}$

Step 11.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Step 12.

1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Add $\frac{1}{3}$ × row 2 to row 1

$$\frac{1}{3} \times 1 + -\frac{1}{3} = 0$$

$$\frac{1}{3} \times \frac{2}{5} + -\frac{1}{3} = -\frac{1}{5}$$

$$\frac{1}{3} \times \frac{11}{10} + \frac{1}{3} = \frac{7}{10}$$

Step 13.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Step 14.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$

Add $-\frac{5}{3}$ × row 2 to row 3

$$-\frac{5}{3} \times 1 + \frac{5}{3} = 0$$

$$-\frac{5}{3} \times \frac{2}{5} + -\frac{1}{3} = -1$$

$$-\frac{5}{3} \times \frac{11}{10} + \frac{4}{3} = -\frac{1}{2}$$

Step 15.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	-1	$-\frac{1}{2}$

Step 16.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	-1	$-\frac{1}{2}$

Pivot at row 3 col 3

Step 17.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	-1	$-\frac{1}{2}$

Multiply row 3 by -1

Step 18.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	1	$\frac{1}{2}$

Step 19.

1	0	$-\frac{1}{5}$	$\frac{7}{10}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	1	$\frac{1}{2}$

Add $\frac{1}{5}$ × row 3 to row 1

$$\frac{1}{5} \times 1 + -\frac{1}{5} = 0$$

$$\frac{1}{5} \times \frac{1}{2} + \frac{7}{10} = \frac{4}{5}$$

Step 20.

1	0	0	$\frac{4}{5}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	1	$\frac{1}{2}$

Step 21.

1	0	0	$\frac{4}{5}$
0	1	$\frac{2}{5}$	$\frac{11}{10}$
0	0	1	$\frac{1}{2}$

Add $-\frac{2}{5}$ × row 3 to row 2

$$-\frac{2}{5} \times 1 + \frac{2}{5} = 0$$

$$-\frac{2}{5} \times \frac{1}{2} + \frac{11}{10} = \frac{9}{10}$$

Step 22. The Reduced Matrix

1	0	0	$\frac{4}{5}$
0	1	0	$\frac{9}{10}$
0	0	1	$\frac{1}{2}$

Solution: $x = \frac{4}{5}$, $y = \frac{9}{10}$, $z = \frac{1}{2}$

17. Curve Balls

For the last two examples, everything went according to plan. We had three equations in three variables and was able to obtain a single (unique) solution $x_1 = a$, $x_2 = b$, $x_3 = c$.
But it turns out that life is not always what we expect.
Like a batter in the major leagues, sometimes you get the pitch you expect, sometimes you are thrown a curve ball.
The next two examples illustrate what can go wrong (if you are looking for a single, unique solution).

18. Example 3

Step 1. The Initial Matrix.

3	−1	1	1
7	1	−1	6
2	1	−1	2

Step 2.

3	−1	1	1
7	1	−1	6
2	1	−1	2

Pivot at row 1 col 1

Step 3.

3	−1	1	1
7	1	−1	6
2	1	−1	2

Multiply row 1 by $\frac{1}{3}$

Step 4.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
7	1	−1	6
2	1	−1	2

Step 5.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
7	1	-1	6
2	1	-1	2

Add $-7 \times$ row 1 to row 2

$$-7 \times 1 + 7 = 0$$

$$-7 \times -\frac{1}{3} + 1 = \frac{10}{3}$$

$$-7 \times \frac{1}{3} + -1 = -\frac{10}{3}$$

$$-7 \times \frac{1}{3} + 6 = \frac{11}{3}$$

Step 6.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{11}{3}$
2	1	-1	2

Step 7.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{11}{3}$
2	1	-1	2

Add $-2 \times$ row 1 to row 3

$$-2 \times 1 + 2 = 0$$

$$-2 \times -\frac{1}{3} + 1 = \frac{5}{3}$$

$$-2 \times \frac{1}{3} + -1 = -\frac{5}{3}$$

$$-2 \times \frac{1}{3} + 2 = \frac{4}{3}$$

Step 8.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 9.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\boxed{\frac{10}{3}}$	$-\frac{10}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Pivot at row 2 col 2

Step 10.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{11}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Multiply row 2 by $\frac{3}{10}$

Step 11.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	1	-1	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 12.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	1	-1	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Add $\frac{1}{3}$ × row 2 to row 1

$$\frac{1}{3} \times 1 + -\frac{1}{3} = 0$$

$$\frac{1}{3} \times -1 + \frac{1}{3} = 0$$

$$\frac{1}{3} \times \frac{11}{10} + \frac{1}{3} = \frac{7}{10}$$

Step 13.

1	0	0	$\frac{7}{10}$
0	1	-1	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 14.

1	0	0	$\frac{7}{10}$
0	1	-1	$\frac{11}{10}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Add $-\frac{5}{3}$ × row 2 to row 3

$$-\frac{5}{3} \times 1 + \frac{5}{3} = 0$$

$$-\frac{5}{3} \times -1 + -\frac{5}{3} = 0$$

$$-\frac{5}{3} \times \frac{11}{10} + \frac{4}{3} = -\frac{1}{2}$$

Step 15. The Reduced Matrix

1	0	0	$\frac{7}{10}$
0	1	-1	$\frac{11}{10}$
0	0	0	$-\frac{1}{2}$

$0 = -\frac{1}{2}$ is impossible

No Solution

19. Example 4

Step 1. The Initial Matrix.

3	-1	1	1
7	1	-1	5
2	1	-1	2

Step 2.

3	-1	1	1
7	1	-1	5
2	1	-1	2

Pivot at row 1 col 1

Step 3.

3	-1	1	1
7	1	-1	5
2	1	-1	2

Multiply row 1 by $\frac{1}{3}$

Step 4.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
7	1	-1	5
2	1	-1	2

Step 5.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
7	1	-1	5
2	1	-1	2

Add $-7 \times$ row 1 to row 2

$$-7 \times 1 + 7 = 0$$

$$-7 \times -\frac{1}{3} + 1 = \frac{10}{3}$$

$$-7 \times \frac{1}{3} + -1 = -\frac{10}{3}$$

$$-7 \times \frac{1}{3} + 5 = \frac{8}{3}$$

Step 6.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$
2	1	-1	2

Step 7.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$
2	1	-1	2

Add $-2 \times$ row 1 to row 3

$$-2 \times 1 + 2 = 0$$

$$-2 \times -\frac{1}{3} + 1 = \frac{5}{3}$$

$$-2 \times \frac{1}{3} + -1 = -\frac{5}{3}$$

$$-2 \times \frac{1}{3} + 2 = \frac{4}{3}$$

Step 8.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 9.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\boxed{\frac{10}{3}}$	$-\frac{10}{3}$	$\frac{8}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Pivot at row 2 col 2

Step 10.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{10}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Multiply row 2 by $\frac{3}{10}$

Step 11.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	1	-1	$\frac{4}{5}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 12.

1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	1	-1	$\frac{4}{5}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Add $\frac{1}{3}$ × row 2 to row 1

$$\frac{1}{3} \times 1 + -\frac{1}{3} = 0$$

$$\frac{1}{3} \times -1 + \frac{1}{3} = 0$$

$$\frac{1}{3} \times \frac{4}{5} + \frac{1}{3} = \frac{3}{5}$$

Step 13.

1	0	0	$\frac{3}{5}$
0	1	-1	$\frac{4}{5}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Step 14.

1	0	0	$\frac{3}{5}$
0	1	-1	$\frac{4}{5}$
0	$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$

Add $-\frac{5}{3}$ × row 2 to row 3

$$-\frac{5}{3} \times 1 + \frac{5}{3} = 0$$

$$-\frac{5}{3} \times -1 + -\frac{5}{3} = 0$$

$$-\frac{5}{3} \times \frac{4}{5} + \frac{4}{3} = 0$$

Step 15. The Reduced Matrix

1	0	0	$\frac{3}{5}$
0	1	-1	$\frac{4}{5}$
0	0	0	0

Solution: $x = \frac{3}{5}$, $y - z = \frac{4}{5}$, $z = z$

$x = \frac{3}{5}$, $y = z + \frac{4}{5}$, $z = z$