Math 210 Lecture Notes: Sections 7.1 -7.2 Probability

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1. Outcomes of an Experiment

A sample space for an experiment is a set of outcomes for the experiment such that

- any two distinct outcomes are disjoint
  (meaning it is impossible for them both to occur)
- and some outcome in the sample space has to occur, no matter what happerns.

Each particular outcome is called a sample point A subset of the sample space is called an event

2. Example

Suppose a die (singular of dice) is rolled once and the number of dots showing on top is observed.

The sample space consists of the six outcomes: $\{1, 2, 3, 4, 5, 6\}$

An event could be the situation that an odd number appears.

This particular event is the subset $E = \{1, 3, 5\}$

Notice that this event E is not one of the sample points.

An event consisting of just one sample point is called a simple event

3. Example

A probability distribution is an assignment of a real number $P(s_j)$ for each sample point s_j in the sample space such that

(i) $0 \leq P(s_j) \leq 1$ and

(ii) the sum of these numbers $P(s_j)$ taken over every point in the sample space is 1.

$P(s_j)$ is called the proababilty that s_j happens.

We define the probability of an event E to be the sum of the probabilities of the individual sample points that constitute that event.

4. Baseball Example

Suppose Alex Rodriguez has an offial at-bat.

The two possible outcomes are: hit or out.

A batter in baseball has other possibilities: a base-on-ball, being hit by the pitch, a sacrifice fly to score a runner from third, a sacrifice bunt to advance a runner. But none of these possibilities are considered part of an offical at-bat.

The assignment of the weights

$$P(hit) = .306 \qquad P(out) = .694$$

represent the fact that over his lifetime (till now) Alex Rodriguez has gotten a hit in 30.6 percent of his official at-bats.

In baseball the number 306 is called a batting average.

5. Average versus Great

What distinguishes an average hitter from a great hitter?

Put differently, how many more hits per week does a great hitter need to get compared with an average hitter?

The answer may surprise you.

Suppose a hitter has 20 official at-bats in one week. This averages to just about 3 per game, which is reasonable considering that walks and sacrifices do not count as official at-bats.

6. Average versus Great

The batting average of an average hitter is, say, 250, which means he gets a hit in 5 out of 20 at-bats:

$$\frac{5}{20} = .250$$

A great hitter, like Alex Rodriguez, has a batting average of, say, 300, which means he gets a hit in 6 out of 20 at-bats.

$$\frac{6}{20} = .300$$

So to raise his average by 50 points an average hitter needs only one more hit per week.

7. Elections

Suppose an election between candidates A and B takes place. We may take as the sample space

A wins   B wins   tie

Without further informations, it is impossible to say what the appropriate probability distribution might be for this space.

The purpose of public opinion polls is to try to estimate reasonable values of the probability distribution.

8. Uniform Probability Space

In a uniform probability space, the probabilities assigned to each sample point are all the same.

If there are n sample points $s_1, s_2, \ldots, s_n$ then for each k between 1 and n:

$$P(s_k) = \frac{1}{n}$$

Such spaces can also be thought of as selecting differently marked balls from a bag.

9. The Bag Model

Imagine a bag (or box) containing balls of various kinds–having various colors for example.

Assume that a certain fraction p of these balls are of type A. This means

- N = total number of balls
- N_{A} = total number of balls of type A
- $p = \dfrac{N_{\mathrm{A}}}{N}$

We draw a ball from the bag and say:
The probability of drawing a ball of type A is

$$P(A) = p.$$

10. Example

If the bag contains red and blue balls,
two-fifths of which are red,

then the probability of drawing a red ball is

two out of five, or

$$P(red) = \frac{2}{5} = .4 = 40\%$$

11. First Principles

The number N_{A} of balls of type A cannot be negative and cannot exceed the total number of balls in the bag. That is,

$$0 \leq N_{\mathrm{A}} \leq N$$

Dividing by N we get

$$\frac{0}{N} \leq \frac{N_{\mathrm{A}}}{N} \leq \frac{N}{N}$$

or

$$0 \leq P(A) \leq 1$$

12. The Negation Rule

If N_{A} counts the number of balls of type A,

then $N - N_{\mathrm{A}}$ counts the number of balls which are *not* of type A.

Thus the probability that a ball chosen at random is not of type A is

$$\frac{N - N_{\mathrm{A}}}{N} = \frac{N}{N} - \frac{N_{\mathrm{A}}}{N} = 1 - \text{ probability of type A}$$

If there is a 40% chance of rain, then there is a 60% chance it won't rain.

13. English versus Set Theory

Since events are technically sets, the event that A does not occur is A^{c}.

We will abuse notation and write $P(\text{not } A)$ when we mean $P(A^{\mathrm{c}})$

Similarly if A and B are events (by definition subsets of the sample space) then we will often write

$P(A \text{ and } B)$ for $P(A \cap B)$

$P(A \text{ or } B)$ for $P(A \cup B)$

Thus the Negation Rule can be stated

$$P(\text{not } A) = 1 - P(A)$$

14. Social Security Numbers

What is the probability that at least two digits of your social security number are the same?

Since there are nine digits in a social security number, and 10 choices for each digit (0 through 9), there are $N = 10^9$ social security numbers.

What proportion of these have duplicate digits?

It turns out that it is much easier to compute the number of social security numbers in which the digits are different.

15. Social Security Numbers

Let N_a denote the number of social security numbers in which two or more digits are alike

and N_d denote the number of social security numbers in which all nine digits are different.

What is N_d?

Answer: $N_d = P(10, 9) = 10 \cdot 9 \cdot 8 \cdot 7 \cdots 3 \cdot 2$

$$P(\text{different}) = \frac{N_d}{N} = \frac{9 \cdot 8 \cdot 7 \cdots 3 \cdot 2}{10^9} = .0036288$$

By the Negation Rule, $P(\text{two digits are alike}) = 1 - P(\text{different}) = .9963712$

16. Reducing Problems to the Bag Model

- Coin Toss: Heads or Tails
- A Single Die
- Rain
- Birth Day
- Gender Selection
- Death
- Cards
- Rolling a Pair of Dice

17. Heads or Tails

- Assuming the coin is "fair."
- The bag contains one ball bearing the inscription "heads"
- and one ball bearing the inscription "tails."
- The probability of heads is $\frac{1}{2}$
- and the probability of tails is $\frac{1}{2}$.

18. A single die

- The plural of die is dice.
- With a fair die the probability of throwing a 3 is $\frac{1}{6}$.
- What does this mean in the bag model?
- The bag contains 6 balls, bearing the inscriptions 1, 2, 3, 4, 5, and 6.
- A throw of the die corresponds to drawing one of the six balls from the bag.
- The probability of throwing a 3 is

$$p = \frac{N_3}{N} = \frac{1}{6}$$

19. Rain

- A weather forecaster says "There is a 40% chance of rain."
- This means put 40 red balls in the bag (red for rain) and
- 60 blue balls (blue for blue skies).
- A cloud choses a ball from the bag.
- A red ball causes the cloud to release precipitation.

20. Birth Day

- The probability of being born on a Monday is 1/7.
- The bag contains slips of paper bearing the names of the days of the week.
- The slip of paper drawn from the bag by the baby-to-be-born determines its birth day.

21. Gender Selection

- The probability of the baby being a boy is $\frac{1}{2}$.
- (Actually it is somewhat higher: 51.4%.)
- On fertilization, the ovum draws a slip of paper from the bag containing the slips marked "boy" or "girl."
- (In effect, the sperm can be considered as bearing the two inscriptions;
- male sperm being somewhat swifter account for the discrepancy from 50%.)

22. Death

- The probability of dying at some time or other is 100%.
- For a Dutchman the probability of dying within one year is 0.77%.
- Such is life: out of 10,000 balls,
- 77 are colored red and 9923 are blue.
- When we meet someone in the street, we draw a ball from the bag.
- A red ball says "dead within a year," a blue ball says "she will live a year longer."

23. Cards

- There are 52 cards in a standard deck of cards (not counting the jokers).
- There are four suits:
  - Diamond $\diamondsuit$
  - Heart $\heartsuit$
  - Club $\clubsuit$
  - Spade $\spadesuit$
- Each suit contains 13 cards:

Ace 2 3 4 $\cdots$ 9 10 Jack Queen King

24. Cards Cont'd

A card is drawn from a well-shuffled deck. The probability that the card is

- the queen of hearts is $\dfrac{1}{52}$

- a heart is $\frac{13}{52} = \frac{1}{4}$
- a queen is $\frac{4}{52} = \frac{1}{13}$
- a heart or a queen is $\frac{?}{52}$

25. A pair of dice

36 throws are conceivable, namely the pairs $(first, second)$ where

- the first die shows $first$ dots and
- the second die shows *second* dots.

26. The Dice Grid

These outcomes could be arranged in a 6 by 6 grid:

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

27. The First Die is 5

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(first die is 5)} = \frac{6}{36} = \frac{1}{6}$$

28. The Second Die is 5

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(second die is 5)} = \frac{6}{36} = \frac{1}{6}$$

29. Both Dice are 5

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(two 5's)} = \frac{1}{36}$$

30. At least one Die is 5

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(at least one 5)} = \frac{11}{36}$$

31. Both dice are the same (doubles)

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(doubles)} = \frac{6}{36} = \frac{1}{6}$$

32. The sum is two (snake-eyes)

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(sum is 2)} = \frac{1}{36}$$

33. The sum is 9

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(sum is 9)} = \frac{4}{36} = \frac{1}{9}$$

34. The sum is 7 (craps)

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(sum is 7)} = \frac{6}{36} = \frac{1}{6}$$

35. Mutually Exclusive Events

Two events are called **mutually exclusive** if they cannot both occur simultaneously.

Examples:

- Catholic and Protestant
- Male and Female
- Heads and Tails

36. Rule for mutually exclusive events

When two events are mutually exclusive, then the probability that one or the other will occur is the sum of the two probabilities.

Balls in a Bag.

There are 5 green, 3 yellow, and 2 red balls in a bag.

The probability that a ball chosen at randon is green or red is the probability the ball is green
+ the probability it is red

$$= \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$$

37. Single throw of a die

A single die is thrown.

The probability the number on top is a 5 or a 6 is

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

38. Non–mutually exclusive events

Two events are called **non mutually exclusive** if it is possible for both to occur.

Examples:

- Catholic and Male
- Lawyer and Liar
- Ace and Club

39. Rule for non mutually exclusive events

When two events are non mutually exclusive, then the probability that at least one of them occurs is the sum of the two probabilities minus the probability that they both occur.

Written as a math formula

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

40. Cards.

A card is drawn from a standard deck.

The probability it is an ace or a club equals

the probability it is an ace + the probability it is club − the probability it is both

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

41. Two dice

A pair of dice is thrown.

The probability that at least one die is a 5 is

prob(first die is 5) + prob(second die is 5)

- prob(both dice are 5)

$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}.$

42. At least one Die is 5

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

$$\text{Prob(at least one 5)} = \frac{11}{36}$$

43. Lawyers and Liars

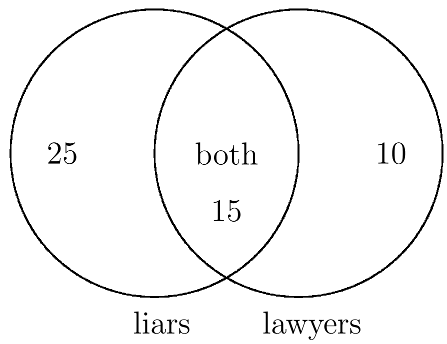
- There are 100 people at a party.
- Forty are liars.
- Twenty-five are lawyers.
- 15 of the lawyers are liars.

If a person is drawn at random at the party (say to win a door prize), what is the probability that he or she is a lawyer or a liar?

By the Formula

$$\text{P(lawyer} \cup \text{liar)} = \text{P(lawyer)} + \text{P(liar) - P(both)} = \tfrac{25}{100} + \tfrac{40}{100} - \tfrac{15}{100} = \tfrac{50}{100} = \tfrac{1}{2}$$

44. Lawyer and Liar Picture



45. binomial probability distribution

P(lawyer or liar)= $\frac{50}{100}$

In a binomial probability distribution, there are two possible outcomes: success or failure.

If p is the probability of success, then $q = 1 - p$ is the probability of failure.

The probability of obtaining exactly k successes out of n attempts is

$$C(n,k)p^k q^{n-k}$$

Such estimates are often used in quality control or error correction.

For example CD technology is able to correct defects on the disk, such as scratches, but only so many. Engineers would like to know the probabilty that say 5% of the information could be corrupted.

46. Dash for Junkers

A car dealer discovers that 8% of the cars received have defective brakes. A shipment of 10 cars arrives.

What is the probability none have defective brakes?

Solution: $p = .92$ is the probability the brakes will be good; $q = .08$ is the probability the brakes will be defective.

The probability all 10 cars will have good brakes is

$(.92)^{10}\,(.08)^0 = .434$

What is the probability exactly 2 have defective brakes?

Solution: By the formula, the probability of 8 good and 2 defective car brakes is

$C(n,k)p^kq^{n-k} \;=\; C(10,2)(.92)^8(.08)^2 \;=\; \dfrac{10\cdot 9}{2\cdot 1}(.92)^8(.08)^2 \;=\; 45(.92)^8(.08)^2$
$= .1478$

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$= .1478$

48. Taking Tests

What is the probability that a 90 percent test taker will score a 90 on a 10 question test?

You might think the answer is “all the time”

But a little thought convinces us she could score below 90 some of the time and above 90 on other tests, which average to 90 for the semester.

$p =$ probability of success $= .9$

$q =$ probability of failure $= .1$

A score of 90% requires 9 correct answers out of 10.

By our formula, the chance of doing this is

$C(n,k)p^k q^{n-k} = C(10,9)(.9)^9(.1)^1 = C(10,1) \times .9^9 \times .1 = 10 \times .9^9 \times .1 = .3874$

So a 90 percent test taker has only a 39 percent chance of score a 90 on a ten question test. That's crazy!

49. Flush in Poker

A 5-card poker hand is called a flush if all 5 cards are of the same suit.

What is the probability that a 5-card poker hand contains all hearts?

The number of 5-card poker hands is $N = C(52,5)$

Remember: the order in which the cards are dealt doesn't matter

How many poker hands consist of all hearts?

Instead of drawing 5 cards from the standard 52–card deck, we select them from the 13 hearts in $N_h = C(13,5)$ ways.

50. All Heart

So the probability of geeting 5 hearts is

$$\text{P(5 hearts)} = \frac{N_h}{N} = \frac{C(13,5)}{C(52,5)}$$

$$= \frac{(13 \cdot 12 \cdot 11 \cdot 10 \cdot 9)/5!}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)/5!}$$

$$= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

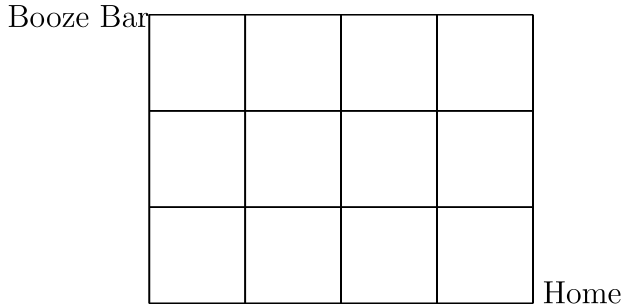
$$= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 3 \cdot 3}{4 \cdot 13 \cdot 17 \cdot 3 \cdot 5 \cdot 10 \cdot 49 \cdot 4 \cdot 12}$$

$$= \frac{11 \cdot 3}{4 \cdot 17 \cdot 5 \cdot 49 \cdot 4} = \frac{33}{66640} \approx \frac{1}{2019.4}$$

The probability of being dealt a 5-card flush is four times this amount, or roughly 1 out of 505.

51. The Drunk

A man leaves a bar which is 4 blocks west and 3 blocks north of his house.



He walks a few steps to the first corner. Suppose that at each of the seven corners that he comes to, he has an equal chance of going straight or turning left or right. What is the probability that he will make it home after walking the seven blocks?

We have seen this problem before.

52. Drunk Problem Solved

The total number of 7-block routes is $N = 3^7$, since he has 3 ways to go for each of the 7 blocks.

In order to get home in 7 blocks,, the drunk needs to go South out for 3 blocks and East for 4 blocks.

The number of ways to choose 3 out of 7 blocks in which to walk South, traveling East on the other 4 blocks, is $C(7,3) = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35$

So the probability of his walking to his home is

$$\frac{C(7,3)}{3^7} = \frac{35}{2187} = .01600$$

The drunk has a 1.6% chance of walking home.

Moral: don't drink and walk.