

MATH 210 LECTURE NOTES:
SECTIONS 6.1 -6.2
SETS AND COUNTING

Richard Blecksmith
Dept. of Mathematical Sciences
Northern Illinois University

1. SETS

- A **set** is a collection of objects.
- The terms “set,” “collection,” and “family” are synonymous.
- If A is a set, then “ $x \in A$ ” means that x is an *element* (or *member* of A , or that x belongs to A).
- The notation $x \notin A$ indicates x is *not* an element of A .
- Sets A and B are *equal*, $A = B$, if and only if they have the same elements.

2. SET NOTATION

- Curly braces are used for set description.
- Sets may be specified by listing,
- for example $\{1, 2, 3\}$, $\{1, 2, 3, \dots\}$.
- If $P(x)$ is a proposition about x , $\{x : P(x)\}$ is the set of exactly those x for which $P(x)$ is true.
- for example $\{x : x \text{ is a positive integer}\}$.
- The *empty set* is denoted by \emptyset .

3. SUBSET

- A set A is a **subset** of a set B if and only if each element of A is also an element of B .
- $A \subseteq B$ means that A is a subset of B .
- $\{a, c\} \subseteq \{a, b, c, d\}$
- the set of even integers is a subset of the set of integers
- the set of primes is not a subset of the set of odd integers Why not?

4. UNIONS

- **Unions** of sets are indicated by “ \cup .”
- Thus $A \cup B$ and $A \cup B \cup C$ denote unions.
- $A \cup B$ is the set of elements belonging to either set A , set B , or both.
- $\{a, c\} \cup \{b, c\} = \{a, b, c\}$
- the set of even integers \cup the set of odd integers is the set of integers

5. INTERSECTIONS

- **Intersections** of sets are indicated by “ \cap ,” with usage analogous to those for “ \cup .”
- $A \cap B$ is the set of elements belonging to both set A and set B .
- $\{a, b\} \cap \{b, c\} = \{b\}$
- the set of even integers \cap the set of odd integers is \emptyset

6. COMPLEMENTS

- Frequently one speaks of **complements** of sets.
- Let U be the universal set for discussion.
- If $A \subseteq U$ is a set, then $A^c = \{x : x \in U \text{ and } x \notin A\}$.
- A^c is called the *complement* of A (relative to the universal set U).
- If $U = \{a, b, c, d, e\}$, then $\{a, c\}^c = \{b, d, e\}$
- If U is the set of integers, then the complement of the set of odd integers is the set of even integers.

7. COMBINING OPERATIONS

If we stick with just one operation, then unions (or intersections) can be performed in all possible orders.

For example,

$$A \cup B \cup C = B \cup A \cup C$$

When we mix unions and intersections, then the order that we write the sets matters. Even the order in which we parenthesize matters.

For example,

$$(A \cup B) \cap C \neq A \cup (B \cap C)$$

8. DISTRIBUTIVE LAWS

The “distributive” rules for multiplication over addition states

$$a \times (b + c) = (a \times b) + (a \times c)$$

For example,

$$3 \times (5 + 7) = (3 \times 5) + (3 \times 7)$$

that is, $3 \times 12 = 15 + 21 = 36$

Note that addition does not distribute over multiplication:

$$3 + (5 \times 7) \neq (3 + 5) \times (3 + 7)$$

that is, $3 + 35 \neq 8 \times 10$

9. DISTRIBUTIVE LAWS FOR SETS

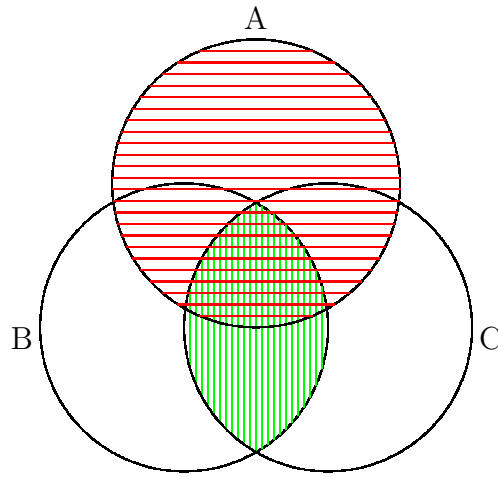
For sets we have the somewhat surprising fact that we can distribute unions over intersections, and also intersections over unions:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and

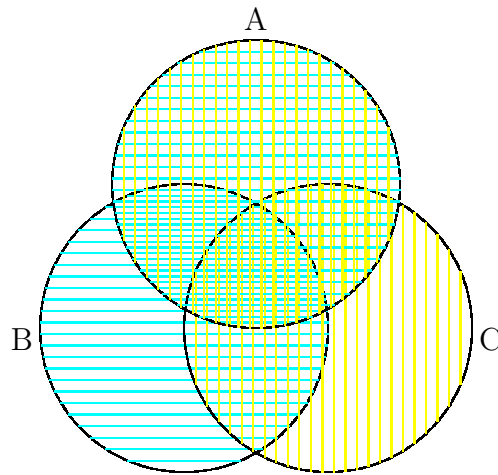
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

10. FIRST VENN DIAGRAM



$$A \cup (B \cap C)$$

11. SECOND VENN DIAGRAM



$$(A \cup B) \cap (A \cup C)$$

12. DEMORGAN'S LAW

The following rules relate complements, unions, and intersections:

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

13. DEMORGAN'S LAW EXAMPLE

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

- $U = \{1, 2, 3, \dots, 8, 9, 10\}$
- $A = \{2, 4, 6, 8, 10\}$
- $B = \{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$
- $(A \cup B)^c = \{7, 9\}$ equals
- $A^c = \{1, 3, 5, 7, 9\}$
- $B^c = \{6, 7, 8, 9, 10\}$
- $A^c \cap B^c = \{7, 9\}$ equals

14. SECOND EXAMPLE

$$(A \cup B)^c = A^c \cap B^c$$

U = people living in U. S.

A = automobile drivers

B = bicycle riders

$A \cup B$ = people who drive or bike (or both)

If you are in $(A \cup B)^c =$
 then you are **not** in $A \cup B$,
 which means
 you do not drive
 and you do not bike.
 that is, you are in $A^c \cap B^c$,

15. SIZE OF A SET

If A is a set, then $n(A)$ denotes the number of elements in set A .

$$n(\{2, 3, 5, 7\}) = 4$$

If $A = \{n : n \text{ is an odd positive integer } \leq 30\}$, then $n(A) = 15$

If P is the set of presidents of the United States, then $n(P) = 43$

Even though Barack Obama is the 44th president, the count is 43, because Grover Cleveland was elected for two non-consecutive terms.

16. RELIGIOUS PREFERENCE

- Out of 2000 people in a community
- 500 are Catholics (set C)
- 270 are Baptists (set B)
- 215 are atheists (set A)
 - $n(A) = 215$
 - $n(B) = 270$
 - $n(C) = 500$
 - What's $n(A \cup B \cup C)$?
 - Answer: $215 + 270 + 500 = 985$

17. LAWYERS AND LIARS

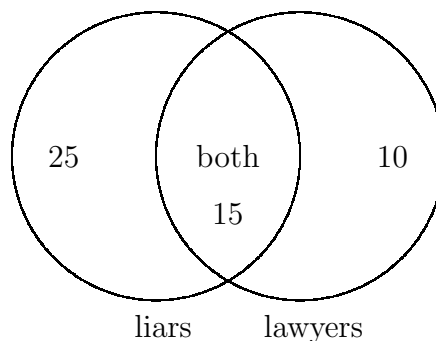
- There are 100 people at a meeting.
- Forty are liars.

- Twenty-five are lawyers.
- If A is the set of liars and B is the set of lawyers, then
 - $n(A) = 40$
 - $n(B) = 25$
 - What's $n(A \cup B)$?
 - Answer: it depends on how many of the lawyers are also liars. Isn't that all of them?

18. LAWYERS AND LIARS CONT'D

- There are 100 people at a meeting.
- Forty are liars.
- Twenty-five are lawyers.
- Suppose we know that 15 of the lawyers are liars.
- $n(\text{lawyers who are not liars}) = n(\text{lawyers}) - n(\text{lawyers who are liars})$
 $= 25 - 15 = 10$
- $n(\text{liars who are not lawyers}) = n(\text{liars}) - n(\text{liars who are lawyers})$
 $= 40 - 15 = 25$

19. LAWYER AND LIAR PICTURE



20. ANOTHER WAY

When we add the number of elements of set A to the number of elements of set B , we are counting the elements of both sets twice.

To compensate, just subtract the number in both, so that it is only counted once.

Counting Principle:

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

21. ANOTHER WAY

A = set of liars

B = set of lawyers

$$n(A) = 40$$

$$n(B) = 25$$

$$n(A \cap B) = 15$$

By the counting principle,

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

$$= 25 + 40 - 15 = 50$$

22. 3 SETS

Things get trickier with three sets.

The idea is to write down three overlapping circles representing the three sets.

This is called a **Venn Diagram**.

Write down all the information about the size of the various intersections.

Fill in the center $n(A \cap B \cap C)$

Next fill in the other intersections.

Now fill in the regions of each of the three sets which do not touch any other set.

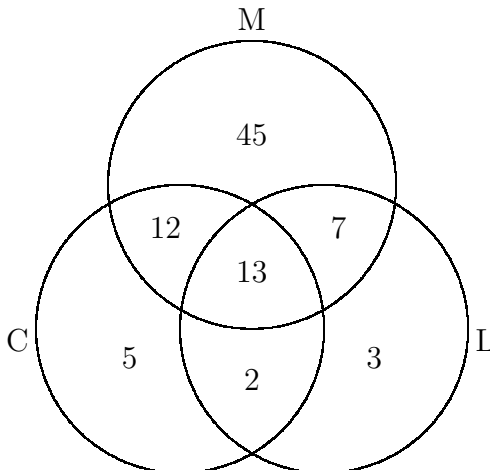
Fill in the border.

23. EXAMPLE

A recent survey at Party–Time U showed that

- 77% tried marijuana
- 32% tried cocaine
- 25% tried LSD
- 25% tried marijuana and cocaine
- 20% tried marijuana and LSD
- 15% tried cocaine and LSD
- 13% tried all three drugs
- What percentage were clean livers (did not try any of the three)?
- What percentage tried marijuana only?

24. THREE SET PICTURE



$$n(M \cap C \cap L) = 13$$

$$n(M \cap C) = 25$$

$$n(M) = 77$$

$$n(M \cap C \cap L^c) = 12$$

$$n(M \cap L^c \cap M^c) = 45$$

$$n(M \cap L) = 20$$

$$n(C) = 32$$

$$n(M \cap L \cap C^c) = 7$$

$$n(C \cap M^c \cap L^c) = 5$$

$$n(C \cap L) = 15$$

$$n(L) = 25$$

$$n(C \cap L \cap M^c) = 2$$

$$n(L \cap M^c \cap C^c) = 3$$

Answers to the questions:

Size of the union of the three sets:

$$n(M \cup C \cup L) = 13 + 7 + 12 + 2 + 45 + 3 + 5 = 87$$

$$n(M \cup C \cup L)^c = 100 - 87 = 13 \text{ clean livers}$$

$$n(M \cap C^c \cap L^c) = 45 \text{ marijuana only}$$