

MATH 211 LECTURE NOTES: CHAPTER 2 GRAPHING

1. MATH 211 BUSINESS CALCULUS

Applications of Derivatives

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2. FINITE INTERVALS

There are four kinds of finite intervals:

Notation	Condition	Name
$[a, b]$	$a \leq x \leq b$	closed
(a, b)	$a < x < b$	open
$[a, b)$	$a \leq x < b$	half-open
$(a, b]$	$a < x \leq b$	half-open

3. INFINITE INTERVALS

There are five kinds of infinite finite intervals:

Notation	Condition	Name
$[a, \infty)$	$x \geq a$	closed ray
(a, ∞)	$x > a$	open ray
$(-\infty, b]$	$x \leq b$	closed ray
$(-\infty, b)$	$x < b$	open ray
$(-\infty, \infty)$	any x	real number line

4. INCREASING VERSUS DECREASING

If I is any one of these nine intervals (finite or infinite) and $f(x)$ is a function defined on I ,

- We say f is increasing on I

if for any two points x_1 and x_2 in I ,

if $x_1 < x_2$ then $f(x_1) < f(x_2)$

In other words, as x values increase, the y values on the graph increase.

- We say f is decreasing on I

if for any two points x_1 and x_2 in I ,

if $x_1 < x_2$ then $f(x_1) > f(x_2)$

In other words, as x values increase, the y values on the graph decrease.

5. RELATION OF DERIV TO GRAPH

function	derivative
increasing	positive
decreasing	negative
horizontal	zero
straight line	constant
steep rising	large positive
gradual rising	small positive
steep falling	large negative
gradual falling	small negative

6. MAJOR THEOREM

Let f be a function which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) .

If $f'(x) > 0$ for every x in (a, b) , then $f(x)$ is increasing on $[a, b]$.

If $f'(x) < 0$ for every x in (a, b) , then $f(x)$ is decreasing on $[a, b]$.

If $f'(x) = 0$ for every x in (a, b) , then $f(x)$ is constant on $[a, b]$.

7. DERIVATIVES

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$\begin{aligned} y' = f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x - 1)(x - 3) \end{aligned}$$

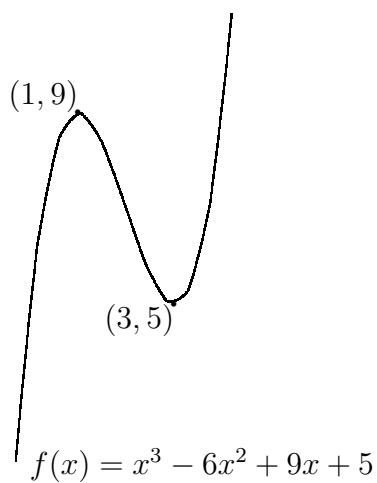
8. DETERMINING SIGN OF f'

$$f'(x) = 3(x - 1)(x - 3)$$

To determine the sign of $f'(x)$ use the following chart

Interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
$x - 1$	neg	pos	pos
$x - 3$	neg	neg	pos
$(x - 1)(x - 3)$	pos	neg	pos
Interpretation	increasing	decreasing	increasing

9. PICTURE OF GRAPH



10. SUBSET OF THE DOMAIN

Let f be a real valued function defined on a set S of real numbers.

The set S is not necessarily the entire domain of f ,

although S must, of course, lie inside the domain of f .

For example, if $f(x) = \sqrt{x}$, then the domain of f is the set of real numbers ≥ 0 .

We could take S to be the closed interval $[1, 2]$.

11. ABSOLUTE MAXIMUM&MINIMUM

The function f has an **absolute maximum** on the set S

if there is at least one point c in S such that

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } S.$$

The number $f(c)$ is called the **absolute maximum value** of f on S .

We say f has an **absolute minimum** on the set S if there is at least one point c in S such that

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } S.$$

12. RELATIVE MAXIMUM&MINIMUM

Let f be a real value function defined on a set S of real numbers.

We say the function f has a **relative maximum** at a point c in S

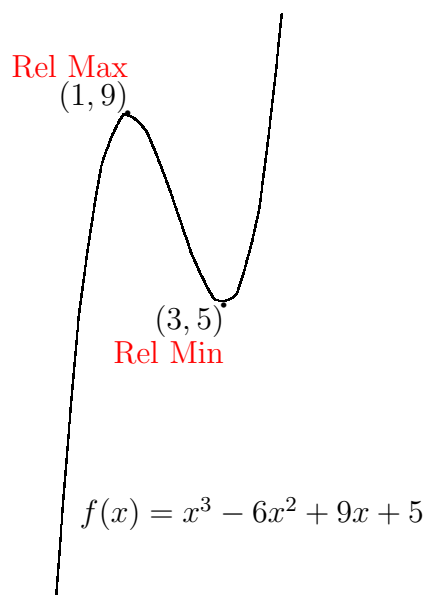
if there is an open interval I containing c such that

$$f(x) \leq f(c) \quad \text{for all } x \text{ lying in both } I \text{ and } S.$$

The concept of **relative minimum** is similarly defined by requiring $f(x) \geq f(c)$ instead of $f(x) \leq f(c)$.

A relative maximum/minimum is sometimes called a local maximum and minimum.

13. RECENT EXAMPLE



14. STRAIGHT LINE EXAMPLE

Consider $f(x) = x$, $S = [0, 2)$.

This function has absolute minimum on S when $x = 0$.

No absolute maximum exists.

This function fails to have an absolute maximum

because $x = 2$ is not in the domain set S .

15. RECIPROCAL EXAMPLE

Consider $f(x) = \frac{1}{x}$, $S = (0, 2]$.

The reciprocal function has absolute minimum on S when $x = 2$.

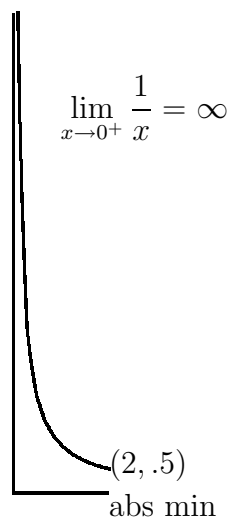
No absolute maximum exists.

This function fails to have an absolute maximum

because it is not continuous at $x = 0$.

that is, because $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

16. GRAPH OF $1/x$



17. VANISHING DERIVATIVE THEOREM

Assume $f(x)$ is a continuous function defined on an open interval I .

Assume that $f(x)$ has a local maximum or minimum at a point c inside I .

If $f'(c)$ exists, then $f'(c) = 0$.

18. FIRST WARNING

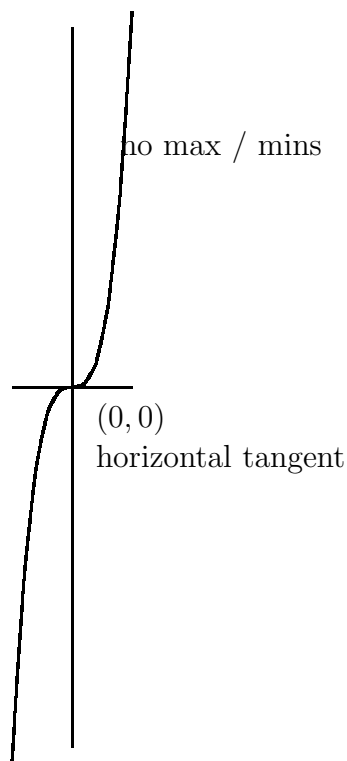
The Vanishing Derivative Theorem does **not** say that if $f'(c) = 0$, then f has a maximum or a minimum at $x = c$.

Example. Consider $f(x) = x^3$.

Here $f'(x) = 3x^2$ and so clearly $f'(0) = 0$,

yet $x = 0$ is neither a local maximum nor a local minimum.

19. GRAPH OF x^3



20. SECOND WARNING

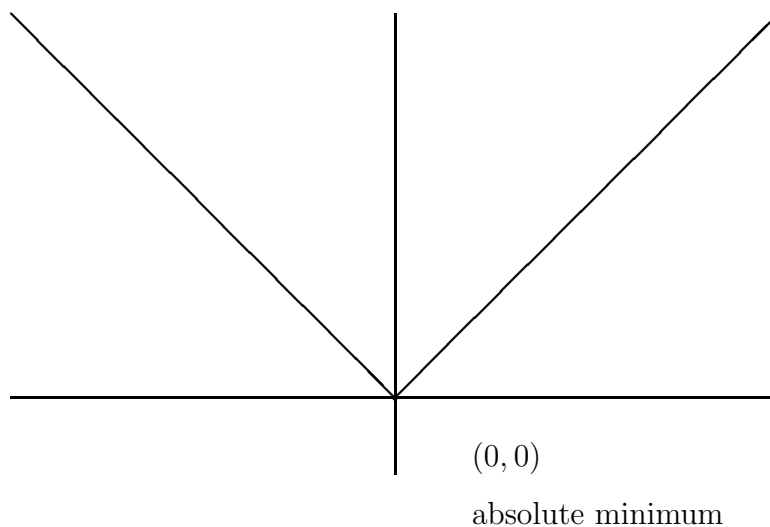
The Vanishing Derivative Theorem does **not** say that if f has a maximum or a minimum at $x = c$, then $f'(c) = 0$.

Example Consider $f(x) = |x|$.

It is clear from the graph that the absolute value function has a relative (in fact, absolute) minimum at $x = 0$,

yet $f'(0)$ is not even defined.

21. GRAPH OF ABSOLUTE VALUE FUNCTION

22. MAX-MIN ^{no derivative} THEOREM

MAX-MIN THEOREM FOR CONTINUOUS FUNCTIONS

Assume $f(x)$ is a continuous function defined on a set S and that $f(x)$ has a local maximum or minimum at a point c in $[a, b]$.

Then we have three possibilities:

(i) $f'(c) = 0$

(ii) $f'(c)$ is undefined

or (iii) c is a boundary point of S

Usually (iii) means that S is an endpoint of an interval.

Points of type (i) or (ii) are called **critical points**.

23. EXAMPLE 1

Find all local and absolute max/min points of $f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2)$$

The critical points are 2 and -2 . Since -2 does not lie in the interval $[0, 4]$, we may ignore it.

The only points that can possibly be relative max / mins are the critical point $x = 2$ and the endpoints 0 and 4.

24. DETERMINING MAX / MINS

$f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3(x - 2)(x + 2)$$

x	$y = x^3 - 12x$	Result
0	0	?
2	$8 - 24 = -16$	Abs Min
4	$64 - 48 = 16$	Abs Max

25. EXAMPLE 2

Find all local and absolute max/min points of $f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 2]$.

$$f'(x) = 3 \cdot \frac{2}{3}x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2$$

$$f'(x) = 0 \iff \frac{2}{x^{1/3}} - 2 = 0$$

$$\iff \frac{2}{x^{1/3}} = 2$$

$$\iff \frac{1}{x^{1/3}} = 1$$

$$\iff x^{1/3} = 1$$

$$\iff x = 1$$

Note that the derivative is undefined when $x = 0$

Why?

The critical points are 0 and 1 and the endpoints are -1 and 2.

26. DETERMINING MAX / MINS

$f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 2]$.

$$f'(x) = \frac{2}{x^{1/3}} - 2$$

x	$y = 3x^{2/3} - 2x$	Result
-1	$3+2 = 5$	Abs Max
0	0	Abs Min
1	$3 - 2 = 1$?
2	.7662	?

27. EXAMPLE 1 REVISITED

Find all relative max/min points of $f(x) = x^3 - 12x$ on the interval $[0, 4]$.

$$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$$

The critical points are 2 and -2 . We ignore -2 since it is not in the interval $[0, 4]$. The endpoints 0 and 4.

We can use the sign of the derivative to determine when the function is going up and going down, thereby isolating relative maximum and minimums.

28. FINDING RELATIVE EXTREMA

Use the chart for $f'(x) = 3(x - 2)(x + 2)$

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x - 2$	neg	neg	pos
$x + 2$	neg	pos	pos
$(x - 2)(x + 2)$	pos	neg	pos
Interpretation	increasing	decreasing	increasing

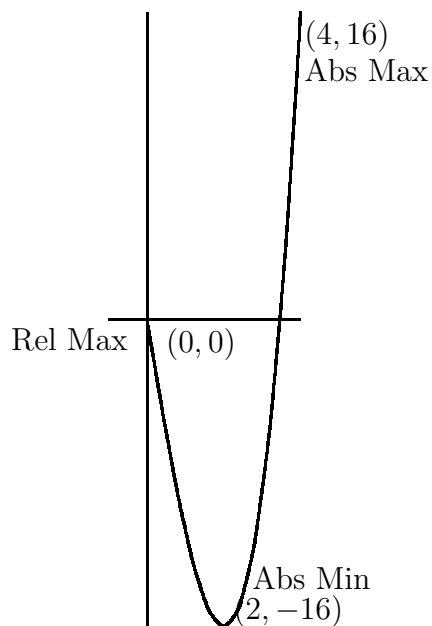
Conclusions:

0 is a relative max since the graph is decreasing to the right of $x = 0$

2 is a relative min since the graph is decreasing to the left of 2 and increasing to the right of 2

4 is a relative max since the graph is increasing to the left of $x = 4$

29. GRAPH OF $x^3 - 12x$



30. FIRST DERIVATIVE TEST

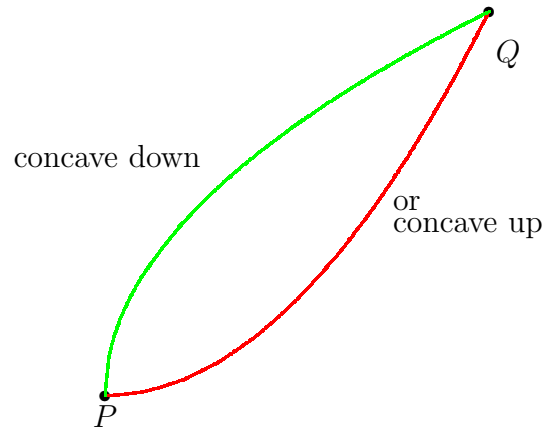
Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and c is a critical point of $f(x)$ in the open interval (a, b)

Interval	(a, c)	(c, b)	Conclusion
Case 1	$y' < 0$	$y' > 0$	MIN
Case 2	$y' > 0$	$y' < 0$	MAX
Case 3	$y' > 0$	$y' > 0$	INCR
Case 4	$y' < 0$	$y' < 0$	DECR

TEST 2 MATERIAL STOPS HERE

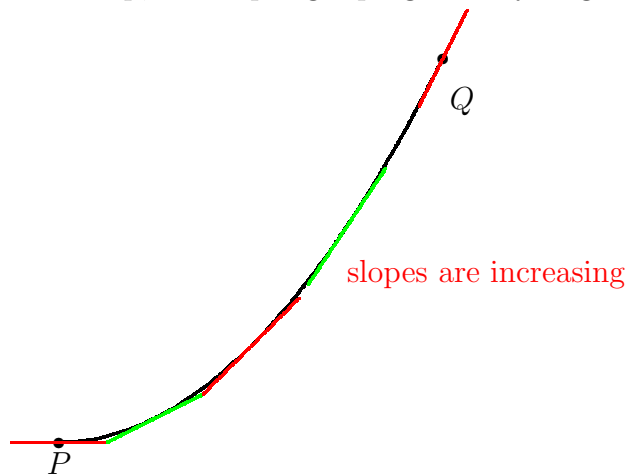
31. SHAPE OF A CURVE

There are two ways that a curve can increase from point P to point Q



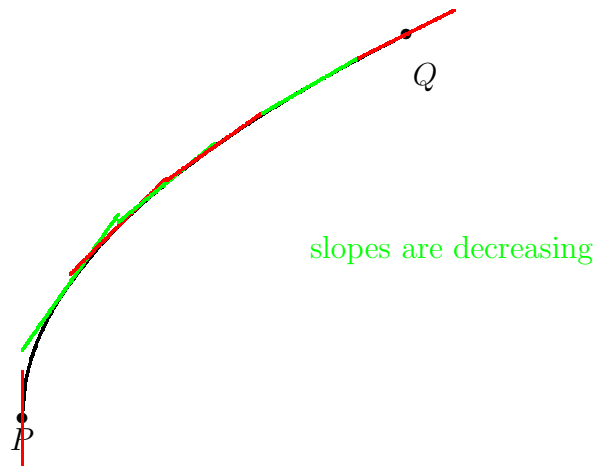
32. CONCAVE UP GRAPHS

When a graph is concave up, the slopes get progressively larger.

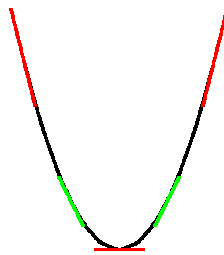


33. CONCAVE DOWN CURVE

There are two ways that a curve can increase from point P to point Q



34. CONCAVE UP CURVE



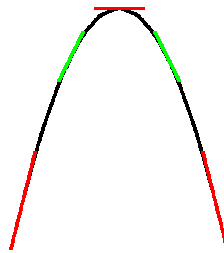
y' is increasing

$$\iff (y')' > 0$$

$$\iff y'' > 0$$

Thus the graph is concave up if and only if $y'' > 0$

35. CONCAVE DOWN CURVE



y' is decreasing

$$\iff (y')' > 0$$

$$\iff y'' < 0$$

Thus the graph is concave down if and only if $y'' < 0$

A point at which concavity changes is called an **inflection point**

36. EXAMPLE - REVISITED

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$y' = f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

Based on this factorization we saw earlier that

Interval	Behavior of graph
$(-\infty, 1)$	increasing
$(1, 3)$	decreasing
$(3, \infty)$	increasing

37. DETERMINING CONCAVITY

$$y = f(x) = x^3 - 6x^2 + 9x + 5$$

$$y' = f'(x) = 3x^2 - 12x + 9$$

$$y'' = f''(x) = 6x - 12 = 6(x - 2)$$

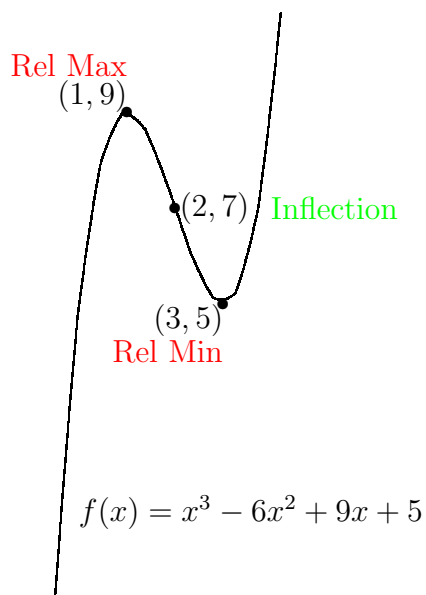
To determine the sign of $f''(x)$ use the following chart

Interval	$(-\infty, 2)$	$(2, \infty)$
$x - 2$	neg	pos
Sign of y''	$y'' < 0$	$y'' > 0$
Interpretation	concave down	concave up

So the graph is concave down to the left of 2 and concave up to the right to 2.

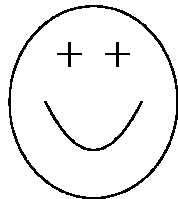
Since the concavity changes at $x = 2$, $(2, f(2))$ is an inflection point. Here,
 $f(2) = 2^3 - 6 \cdot 2^2 + 9 \cdot 2 + 5 = 8 - 24 + 18 + 5 = 7$

38. PICTURE OF GRAPH



39. TWO FACES

Happy Face



concave up mouth

++ eyes

pos second deriv

Sad Face



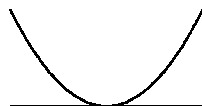
concave down mouth

-- eyes

neg second deriv

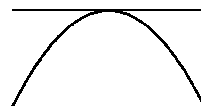
40. MAX VERSUS MIN

Concavity gives us a way to tell maximums from minimums when dealing with critical points



Relative Min
concave up

$$y'' > 0$$



Relative Max
concave down

$$y'' < 0$$

41. SECOND DERIVATIVE TEST

Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and c is a critical point of $f(x)$ in the open interval (a, b) for which $f'(c) = 0$

$f''(c)$	Conclusion
pos	Minimum
neg	Maximum
zero	Don't Know

42. TEST FOR $x^3 - 12x$

$$f(x) = x^3 - 12x$$

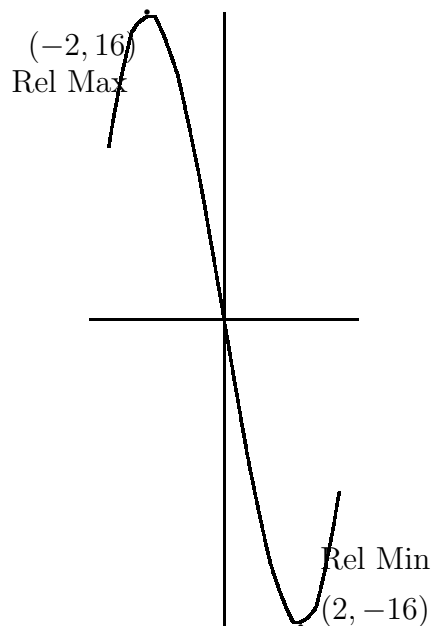
$$f'(x) = 3x^2 - 12$$

$$= 3(x - 2)(x + 2)$$

Critical points: 2 and -2

$$f''(x) = 6x$$

x	$f(x) = x^3 - 12x$	$f''(x) = 6x$	Sign	Concl
-2	$-8 + 24 = 16$	-12	neg	Max
2	$8 - 24 = -16$	12	pos	Min

43. GRAPH OF $x^3 - 12x$ 

44. WARNING

No conclusion can be made if $f''(c) = 0$ as the following three examples show.

For each of these three functions,

$$f'(0) = 0 \text{ and } f''(0) = 0$$

$f(x)$	$f'(x)$	$f''(x)$	Behavior
x^4	$4x^3$	$12x^2$	Min
$-x^4$	$-4x^3$	$-12x^2$	Max
x^3	$3x^2$	$6x$	Neither

For $f(x) = x^4$, $x = 0$ is a minimum

For $f(x) = -x^4$, $x = 0$ is a maximum

For $f(x) = x^3$, $x = 0$ is neither a max nor min

45. SECOND WARNING

The statement

inflection points occur when the second derivative is zero

is slightly misleading.

Although this statement is true in practice for the functions you are likely to see in this course.

Consider the function $f(x) = x^4$.

Here $f''(x) = 12x^2 = 0$ at $x = 0$

But $x = 0$ is not an inflection point, because

This function is always concave up.

It never changes concavity.

46. STRATEGY FOR SKETCHING GRAPHS

Step 1. [Derivatives] Find $f'(x)$ and $f''(x)$

Step 2. [Domain and symmetry] Note the domain of $f(x)$ and look for possible symmetries:

(a) $f(-x) = f(x)$ implies symmetry about y -axis

(b) $f(-x) = -f(x)$ implies symmetry about origin

Step 3. [Critical points] Determine critical points where the derivative is zero or undefined. These are candidates for max/min points.

Step 4. [Increasing/Decreasing] Determine the sign of the derivative in the intervals between successive critical points.

(a) $y' > 0$ means graph is increasing

(b) $y' < 0$ means graph is decreasing

(c) Identify the maximum versus the minimums using the first derivative test.

47. SKETCHING GRAPHS CONTINUED

Step 5. [Inflection points] Determine when $f''(x) = 0$. These are candidates for inflection points.

Step 6. [Concavity] Determine the sign of the second derivative in the intervals between successive points where $f''(x) = 0$.

- (a) $y'' > 0$ means graph is concave up
- (b) $y'' < 0$ means graph is concave down
- (c) Inflection points occur when concavity changes

Step 7. [Table of Points] Make a table of the y values of the critical and inflection points.

Step 8. [Complete the sketch] Connect the points in your graph using the information about increasing/decreasing and concavity.