Applications of Derivatives
Asymptotes and Optimization

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1. Asymptotes

A nonvertical line with equation $y = mx + b$ is called an asymptote of the graph of $y = f(x)$ if the difference $f(x) - (mx + b)$ tends to 0 as x takes on arbitrarily large positive values or arbitrarily large negative values.

If $m = 0$ then $y = b$ is called a horizontal asymptote.

Otherwise $y = mx + b$ is called a slant asymptote.

A verticql line $x = a$ is called a vertical asymptote if $|f(x)|$ takes arbitrarily large values as $x \to a$ from the right or the left.

2. Example

Let $y = f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} = 1 - x^{-2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

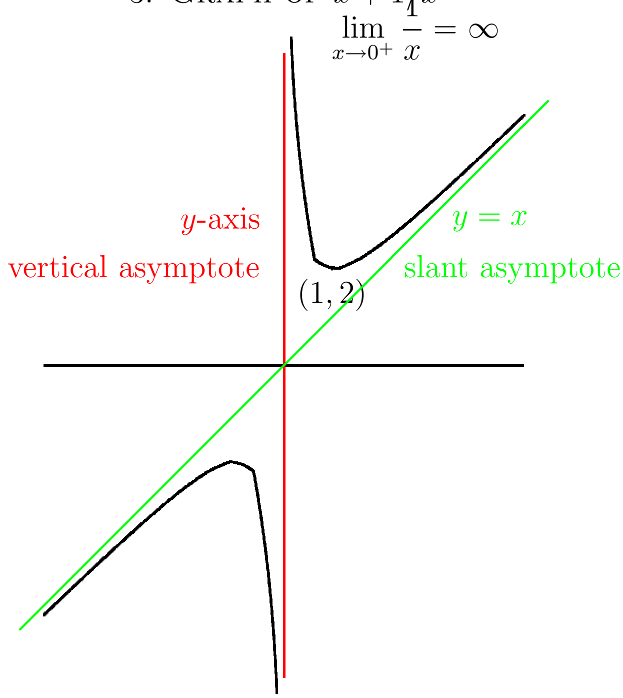
Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + \dfrac{1}{x} = \infty$

$x = 0$ (the y-axis) is a vertical asymptote.

Since $\lim_{x\to\infty} f(x) - x = \lim_{x\to\infty} \dfrac{1}{x} = 0$

the line $y = x$ is a slant asymptote.

3. Graph of $x + 1/x$



4. Vertical Asymptotes

To find vertical asymptotes

look for values of x which make the denominator zero

Example: Consider the function

$$f(x) = \frac{5x^3 + 4x - 8}{x^3 - 2x^2 - 3x}$$

Find the vertical asymptotes of $f(x)$

Solution: Factor the denominator:

$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3) = x(x - 3)(x + 1)$

So vertical asymptotes are:

$x = 0$, $x = 3$, $x = -1$

5. Horizontal Asymptotes

To find horizontal asymptotes

compute the limit as $x \to \infty$ (or $-\infty$)

Example: Consider the function

$$f(x) = \frac{5x^3 + 4x - 8}{x^3 - 2x^2 - 3x}$$

Find the horizontal asymptotes of $f(x)$

Solution: Compute the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5x^3 + 4x - 8}{x^3 - 2x^2 - 3x} \; \frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty} \frac{5 + 4/x^2 - 8/x^3}{1 - 2/x - 3/x^2} = \frac{5 + 0 - 0}{1 - 0 - 0} = 5$$

So the horizontal asymptote is: $\quad y = 5$

6. Computing Limits

When finding the limit as $x \to \infty$ of fractions of polynomials, the rule is

Multiply the fraction by $\dfrac{1/x^m}{1/x^m}$, where

m is the degree of the polynomial in the denominator.

and use the fact that for any positive integer k,

$$\lim_{x\to\infty} \frac{1}{x^k} = 0$$

The same idea applies when computing limits where $x \to -\infty$.

7. Example 1.

Compute the limit

$$= \lim_{x\to\infty} \frac{x^2+8}{x^3-4x} \; \frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty} \frac{1/x - 8/x^3}{1-4/x^2}$$

$$= \frac{0-0}{1-0}$$

$$= 0$$

8. Second example

This limit may end up being infinity

Example: Compute the limit

$$= \lim_{x\to\infty} \frac{2x^4+x}{x^2+5} \; \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{2x^2+1/x}{1+5/x^2}$$

$$= \frac{\infty+0}{1+0}$$

$$= \infty$$

9. Graphs with asymptotes

Sketch the graph of $y = f(x) = \dfrac{x-2}{x+1}$

We need the derivatives

By the quotient rule

$$f'(x) = \frac{(1)(x+1)-(x-2)(1)}{(x+1)^2}$$

$$= \frac{x+1-x+2}{(x+1)^2}$$

$$= \frac{3}{(x+1)^2}$$

By the power rule
$f''(x) = -6(x+1)^{-3}$

10. Graphs continued

Since the first derivative $f'(x) = \dfrac{3}{(x+1)^2}$ is always positive, the function is always increasing.

Since the second derivative $f''(x) = -\dfrac{6}{(x+1)^3}$ is $\begin{cases} \text{negative} & \text{if } x > -1 \\ \text{positive} & \text{if } x < -1 \end{cases}$,

the graph is $\begin{cases} \text{concave down} & \text{if } x > -1 \\ \text{concave up} & \text{if } x < -1 \end{cases}$,

There are no critical points and no inflection points.

11. Find the Asymptotes

For $y = f(x) = \dfrac{x-2}{x+1}$

Vertical asymptotes occur when the denominator $x+1=0$

or when $x=-1$

Note that $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x-2}{x+1} = \frac{-3}{0^+} = -\infty$

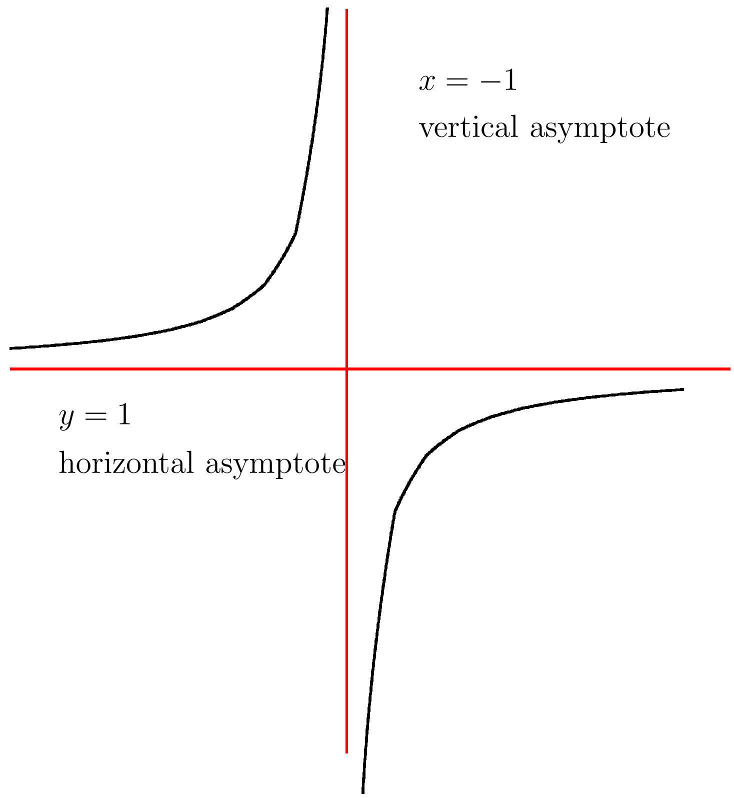
To find the horizontal asymptote(s), compute the limit

$$= \lim_{x\to\infty} \frac{x-2}{x+1}\,\frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1-2/x}{1+1/x} = \frac{1-0}{1+0} = 1$$

Vertical asymptote: $x=-1$

Horizontal asymptote: $y=1$

12. Graph of $(x-2)/(x+1)$



13. Business Applications

Three Inportant functions:

- Revenue $R(x)$
- Profit $P(x)$
- Cost $C(x)$

Theire derivatives are called

- Marginal Revenue $= R'(x)$
- Marginal Profit $= P'(x)$
- Marginal Cost $= C'(x)$

14. Business Applications

The goal of business management is to

- Maximize Profit
- Maximize Revenue
- Minimize Cost

Since maximums and minimums occur when the derivative is zero, our goal is to find values for which

marginal revenue, marginal profit, marginal cost = 0.

15. Maximize Revenue

If the price for a product is p, the revenue (in thousands of dollars) is approximated by

$$R = -0.05p^2 + 0.98p + 18$$

What price maximizes revenue?

Solution:

$R'(p) = -0.1p + 0.98 = 0$

$\iff 0.1p = 0.98$

$\iff p = 9.8$

Question: How do we know that $p = 9.8$ gives a maximum (and not a minimum)?

We are not usually interested in minimizing revenue

unless you are hoping to get government bailout money.

16. Max versus Min

$R = -0.05p^2 + 0.98p + 18$

$R'(p) = -0.1p + 0.98 = 0$

$\iff p = 9.8$

To tell whether this critical point is a max or a min,

Use the second derivative test

$R''(p) = -0.1$

And a negative second derivative implies

$p = 9.8$ is a Maximum

17. Maximize Revenue

The revenue for selling x thousand units of a product can be approximated by

$$R = x^3 - 21x^2 + 120x + 500$$

What value of x maximizes revenue?

Solution:

$R'(x) = 3x^2 - 42x + 120$

$\quad = 3(x^2 - 14x + 40)$

$\quad = 3(x-)(x - 10)$

The critical points are

$x = 4$ and $x = 10$

Question: How do we know whether $p = 4$ and $x = 10$ gives a maximum (and not a minimum)?

18. Max versus Min

$R = x^3 - 21x^2 + 120x + 500$

$R'(x) = 3x^2 - 42x + 120$

Critical points: 4 and 10

$R''(x) = 6x - 42$

Use the second derivative test

x	$R(x)$	$R''(x) = 6x - 42$	Concl
4	708	$24 - 42 = -18$	Max
10	600	$60 - 42 = 18$	Min

Revenue is maximum at $x = 4$

19. Maximize Profit

As a general rule

$P(x) = R(x) - C(x)$

By calculus we know

$P'(x) = R'(x) - C'(x)$

and

$P''(x) = R''(x) - C''(x)$

Thus $P'(x) = 0$ if and only if

$\boxed{R'(x) = C'(x)}$

This critical point is a maximum

if and only if $P''(x) < 0$

if and only if

$\boxed{R''(x) < C''(x)}$

20. Maximize Profit Example

The revenue for selling x thousand units of a product is approximated by

$$R(x) = -0.05x^2 + 2x + 60$$

and the cost for producing x thousand units is approximated by

$$C(x) = 1.5x + 20$$

What level of sales maximizes profit?

Solution: $R'(x) = -0.1x + 2 \quad$ and $\quad C'(x) = 1.5$

Setting $R'(x) = C'(x)$ gives

$-0.1x + 2 = 1.5 \iff -0.1x = -.5 \iff x = 5$

This is a maximum because $R''(x) = -.1$ is less than $C''(x) = 0$

21. Ticket Sales Problem

- A movie multiplex sells tickets for \$8.
- On Friday evenings, it averages 720 tickets sold.
- Each patron spends an average of \$2 on concessions.
- A survey shows that for each \$0.25 drop in ticket price, 20 more people will buy tickets on Friday evenings.
- Assuming that these 20 additional customers also average \$2 in concession sales,
- what ticket price will maximize revenue?

22. Ticket Sales Setup

Let $x =$ the number of 25 cent reductions in ticket price.

The price for a ticket is $p = 8.00 - 0.25x$

The number of customers is $N = 720 + 20x$

The revenue from movie sales is

$pN = (8.00 - 0.25x)(720 + 20x)$

The revenue from concessions is

$2N = 2(720 + 20x)$

The total revenue is

$$
\begin{aligned}
R(x) &= (8.00 - 0.25x)(720 + 20x) + 2(720 + 20x) \\
&= (10 - 0.25x)(720 + 20x)
\end{aligned}
$$

23. Ticket Sales Solution

$R(x) = (10 - 0.25x)(720 + 20x)$

By the product rule,

$$\begin{aligned} R'(x) &= (-0.25)(720 + 20x) + (10 - 0.25x)(20) \\ &= -\tfrac{1}{4} \cdot 720 - \tfrac{1}{4} \cdot 20x + 10 \cdot 20 - (0.25)(20)x \\ &= -180 - 5x + 200 - 5x \\ &= 20 - 10x \end{aligned}$$

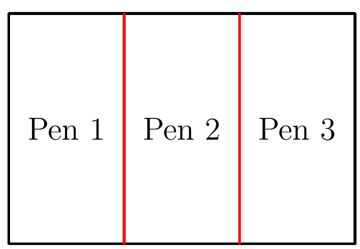
So we see that $R'(x) = 20 - 10x = 0$

if and only if $x = 2$

The movie multiplex should reduce the ticket price by 50 cents to maximize revenue.

24. Dog Kennel Problem

The owner of a kennel has 90 feet of fencing available to enclose three pens as shown



What dimensions maximize the total area of the three pens?

25. Dog Kennel Setup

Let x = length of the pen and y = width

We need $2x$ feet of fencing for the top and bottom

and $4y$ feet of fencing for sides (including the two red dividers)

Since only 90 feet of fencing is available,

$$2x + 4y = 90$$

The problem is to maximize the area

$$A = x \cdot y$$

26. Dog Kennel Solution

Solve

$$2x + 4y = 90$$

for x:

$2x = 90 - 4y$

$x = 45 - 2y$

Subsitute $x = 45 - 2y$ into $A = xy$

$A = xy = (45 - 2y)y = 45y - 2y^2$

Differentiate (with respect to y) and set to 0:

$\dfrac{dA}{dy} = 45 - 4y = 0$

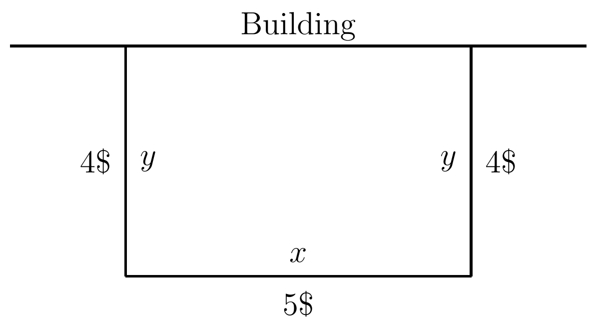
if and only if $y = 45/4 = 11.25$

$x = 45 - 2y = 45 - 22.5 = 22.5$

The dimensions are 11.25 by 22.5

27. Another Fencing Problem

The owner of a long building wishes to attach a rectangular fence as shown



The sides that make up the width cost 4$ per foot
The side that makes up the length costs 5$ per foot

The area must be 810 square feet

What dimensions minimize the cost?

28. Fencing Problem Setup

Let x = length of the pen and y = width

The two side pieces cost $4y$ each

and the one length piece costs $5x$

The total cost is

$$C = 2 \cdot 4y + 5x = 8y + 5x$$

The area is $A = xy$

The problem is to minimize the cost function C

subject to the constraining equation

$$x \cdot y = 810$$

29. Fence Solution

Solve $xy = 810$ for y:

$y = 810/x$

Substitute $y = 810/x$ into the cost equation

$$C = 8y + 5x = 8 \cdot \frac{810}{x} + 5x = \frac{6480}{x} + 5x$$

Differentiate (with respect to x) and set to 0:

$$\frac{dC}{dx} = -\frac{6480}{x^2} + 5 = 0$$

if and only if $\frac{6480}{x^2} = 5$

if and only if $5x^2 = 6480$

if and only if $x^2 = 1296$

Taking the square root $x = \sqrt{1296} = 36$

and $y = 810/x = 810/36 = 22.5$

30. Limit Def Revisited

Recall the limit definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This means that if h is small, then $f'(x)$ is a good approximation of the ratio

$$\frac{f(x+h) - f(x)}{h}$$

In particular, if we let $h = 1$ which is sort of small

$f'(x)$ approximates $f(x+1) - f(x)$

31. Marginal Revenue, Profit, Cost

Applying the statement

$f'(x)$ approximates $f(x+1) - f(x)$

to the functions $R(x)$, $P(x)$, $C(x)$ gives us

- Marginal Revenue $R'(x)$ approximates $R(x+1) - R(x)$
- Marginal Profit $P'(x)$ approximates $P(x+1) - P(x)$
- Marginal Cost $C'(x)$ approximates $C(x+1) - C(x)$

If you have produced x items, then $C(x+1) - C(x)$ is the cost required to produce one more item. Likewise for $P(x+1) - P(x)$ and $R(x+1) - R(x)$

32. Marginal Example

Flying Corn Motors has a weekly profit in dollars of

$$P(x) = -0.006x^3 - 0.2x^2 + 900x - 1200$$

and currently 50 cars are sold per week.

What is the current weekly profit?

Answer: $P(50) = -0.006(50)^3 - 0.2(50)^2 + 900(50) - 1200$

$= -0.006(125{,}000) - 0.2(2500) + 900(50) - 1200$

$= -750 - 500 + 45000 - 1200 = 42{,}550$

What would the profit be if they sold one more car?

Answer: $P(51) = -0.006(51)^3 - 0.2(51)^2 + 900(51) - 1200$

$= -0.006(132{,}651) - 0.2(2601) + 900(51) - 1200$

$= -795.91 - 520.2 + 45900 - 1200 = 43{,}383.89$

33. Marginal Example Continued

The exact increase in profit is

$P(51) - P(50) = 43{,}383.89 - 42{,}550 = 833.89$

What is the marginal profit?

Differentiate

$$P(x) = -0.006x^3 - 0.2x^2 + 900x - 1200$$
$$P'(x) = -0.018x^2 - 0.4x + 900$$

What is the marginal profit when $x = 50$?

Answer:

$P'(50) = -0.018(50)^2 - 0.4(50) + 900 = -0.018(2500) - 0.4(50) + 900 = -45 - 20 + 900 = 835$

Compare 835 with actual increase in profit 833.89