Math 211 Lecture Notes: Sections 1.4 - 1.6

1. Math 211 Business Calculus

Derivatives

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2. Definition of Derivative

The derivative of the function $f(x)$ is defined to be

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

The prime in the symbol $f'(x)$ signifies the derivative of the function $f(x)$

Read $f'(x)$ as "the derivative of f at x" or as "f–prime of x"

Sometimes $f'(x)$ is called the derived function

3. Some Observations

The definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

requires us to

- find $f(x+h)$
- find the difference $f(x+h)-f(x)$
- find the quotient $\dfrac{f(x+h)-f(x)}{h}$
- evaluate the limit of this quotient as $h \to 0$

4. An Example

Compute the derivative $f'(3)$ for the function $f(x) = x^2 + 5x - 11$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\big((3+h)^2 + 5(3+h) - 11\big) - \big(3^2 + 15 - 11\big)}{h}$$

$$= \lim_{h\to 0} \frac{(9 + 6h + h^2 + 15 + 5h - 11) - 13}{h}$$

$$= \lim_{h\to 0} \frac{11h + h^2}{h}$$

$$= \lim_{h\to 0} \frac{(11+h)h}{h}$$

$$= \lim_{h\to 0} 11 + h = 11$$

5. So What

What information do we get from knowing that the derivative of the function $f(x) = x^2 + 5x - 11$ at $x = 3$ has the value 11?

What does the 11 tell us?

Answer 1. Relative Rate of Change.
It tells us that when $x = 3$, the y-values are increasing exactly 11 times as fast as the x-values.

Answer 2. Tangent Line
It tells us that on the graph of $y = f(x)$, the tangent line to the curve at the point $(3, 13)$ has a slope of exactly 11.

Answer 3. Information about the graph of $y = f(x)$. Since the y-values are increasing, the curve must be going upward at the point $(3, 13)$

6. Derivative of a constant function

Suppose $f(x) = c$, a constant.

To find $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= \boxed{0} \end{aligned}$$

7. Derivative of $f(x) = x$

Let $f(x) = x$.

To find $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= \boxed{1} \end{aligned}$$

8. Derivative of x^2

By the definition of derivative, for $f(x) = x^2$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{(2x+h)h}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x + 0 = \boxed{2x}
\end{aligned}$$

9. Expanding cubes

We will need to expand $(x+h)^3$

$$\begin{aligned}
(x+h)^3 &= (x+h)(x+h)^2 \\
&= (x+h)(x^2 + 2xh + h^2) \\
&= \begin{matrix} x^3 + 2x^2h & + xh^2 & \\ x^2h & + 2xh^2 & + h^3 \end{matrix} \\
&= x^3 + 3x^2h + 3xh^2 + h^3
\end{aligned}$$

10. Derivative of x^3

By the definition of derivative, for $f(x) = x^3$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2 + 3x \cdot 0 + 0^2 = \boxed{3x^2}$$

11. Table of Derivatives

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$
x^6	$6x^5$

POWER RULE: the derivative of x^n is nx^{n-1}

12. Warning

- Knowing the power rule does not mean you know calculus
- Any more than knowing the quadratic formula means you know algebra
- But it is a start and . . .
- It is safe to say that you cannot know calulus if you do not know the power rule

13. Time Out for Notation Update

- It is awkward to say "the derivative of x^n is nx^{n-1}"

- Using the prime notation for derivatives, we could write
- $\left(x^n\right)' = nx^{n-1}$"
- But most mathematicians prefer to use an alternative notation due to the Leibnitz, one of the co-inventors of calculus, along with
- Issac Newton, who favored the prime notation

14. Leibnitz Notation

- In the Leibnitz notation, if $y = f(x)$, then the derivative of $f(x)$ is written $\boxed{\dfrac{dy}{dx}}$
- or sometimes $\dfrac{d}{dx}y$
- or even $\dfrac{d}{dx}f(x)$
- Thus the power rule can be expressed as

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

15. Which Notation should you use

- As a rule of thumb, the prime notation is best when working with generic functions: f, g, etc.
- The Leibnitz notation is best when working with letters which represent physical quantities.
- For example, in the formula $A = \pi r^2$
  - A stands for area (of a circle)
  - r stands for radius
  - π is the number 3.1415926 . . .
- The formula $\dfrac{dA}{dr} = 2\pi\, r$ tells us
- the area is increasing by a factor of 2π times the radius

16. Negative and Fractional Powers

Many important function in algebra can be written as powers if we allow negative exponents or fractional exponents.

For example

$$\frac{1}{x} = x^{-1}$$

$$\frac{1}{x^2} = x^{-2}$$

$$\sqrt{x} = x^{1/2}$$

17. New Uses for the Power Rule

The good news is that the power rule still holds true when the exponent is negative or a fraction. Thus,

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}x^{-2} = (-2)x^{-3}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$$

18. A constant times a function

Let $f(x)$ be a function whose derivative we know.

Suppose $g(x)$ is the function obtained by multiplying $f(x)$ by a constant c.

To determine the derivative of $g(x) = c \cdot f(x)$:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\
&= \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h} \\
&= c \cdot \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} \\
&= c \cdot f'(x)
\end{aligned}$$

19. Examples

$$\frac{d}{dx}7x^2 = 7 \cdot 2x = 14x$$

$$\frac{d}{dx}6x^9 = 6 \cdot 9x^8 = 54x^8$$

$$\frac{d}{dx} - 5x^{11} = -5 \cdot 11x^{10} = -55x^{10}$$

With practice you will be able to do these in your head

20. Addition Rule

Let $f(x)$ and $g(x)$ be two functions whose derivatives we know.

To determine the derivative of $p(x) = f(x) + g(x)$:

$$p'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x))}{h}$$

$$= f'(x) + g'(x)$$

21. Summary of Rules

- Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
- Constant Rule: $\frac{d}{dx}c = 0$
- Constant Times Rule: $\frac{d}{dx}\, c \cdot f(x) = c \cdot f'(x)$

- Addition Rule:
  $$\frac{d}{dx}\Big[f(x)+g(x)\Big]=f'(x)+g'(x)$$
- Subtraction Rule: $\frac{d}{dx}\Big[f(x)-g(x)\Big]=f'(x)-g'(x)$

22. Polynomials

Together these rules enable us to differentiate all polynomials.

$\frac{d}{dx}7x^3-5x^2+8x-4$

$=21x^2\ -10x\ +8$

$\frac{d}{dx}11x^5+13x^4-6x^3+12x^2+x-17$

$=55x^4\ +52x^3\ -18x^2\ +24x\ +1$

23. Lines

The general formula for a line is

$$y=mx+b$$

where m is the slope

and b is the y-intercept.

If we know the slope m and a particular point (x_1, y_1) on the line (not necessarily the y-intercept), then the formula for the line is

$$y-y_1=m(x-x_1)$$

This is called the point–slope formula

24. Example

Find the equation of the line passing through the point $(3, 7)$ with slope 5.

Solution: Using the point slope formula

$$y-y_1=m(x-x_1)$$

with $m = 5$ and $(x_1, y_1) = (3, 7)$:

$y - 7 = 5(x - 3)$

$y - 7 = 5x - 15$

$y - 7 + 7 = 5x - 15 + 7$

$y = 5x - 8$

25. Tangent Lines

Find the equation of the tangent line to the curve $y = f(x) = 2x^3 - 5x^2 - 16x + 32$ at the point $(3, f(3))$.

First, plug in $x = 3$ into the function to determine y

$y = f(3) = 2 \cdot 3^3 - 5 \cdot 3^2 - 16 \cdot 3 + 32$

$= 2 \cdot 27 - 5 \cdot 9 - 16 \cdot 3 + 32$

$= 54 - 45 - 48 + 32 = -7$

Next, to find the slope, we need the derivative:

$f'(x) = 6x^2 - 10x - 16$

Plug $x = 3$ into $f'(x)$ to find the slope:

$m = f'(3) = 6 \cdot 3^2 - 10 \cdot 3 - 16$

$= 6 \cdot 9 - 10 \cdot 3 - 16$

$= 54 - 30 - 16 = 8$

26. Tangent Line Continued

So the problem of finding the tangent line reduces to:

Find the equation of the line through the point $(3, -7)$ with slope 8.

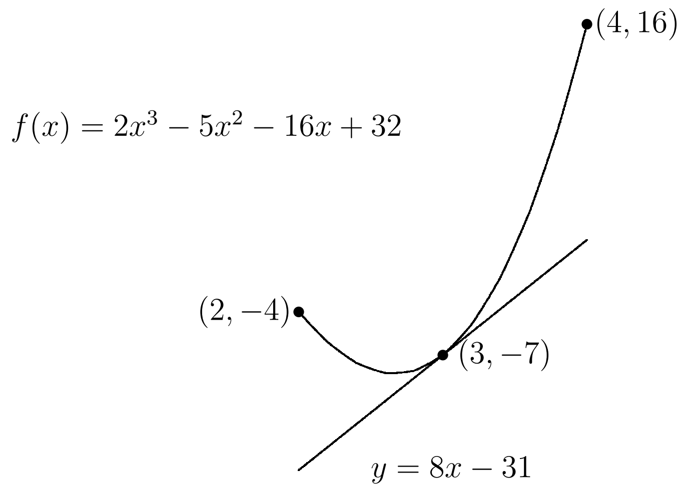
Plug $m = 6$ and $(x_1, y_1) = (3, -7)$ into the point-slope formula:

$y - (-7) = 8(x - 3)$

$y + 7 = 8x - 24$

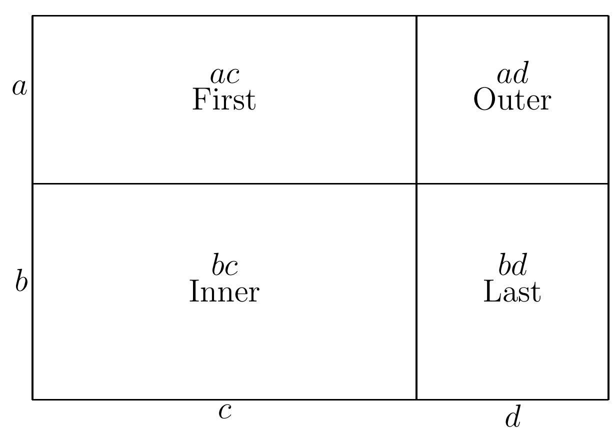
$y = 8x - 31$

27. Picture of Graph



28. Picture of FOIL

$$(a + b) \cdot (c + d) \qquad = ac \ + ad \ + bc \ + bd$$



29. Impressing your Friends

You can use FOIL to calculate products in your head, thereby impressing friends and strangers.

Multiply

$$32 \times 53$$

By FOIL this product is

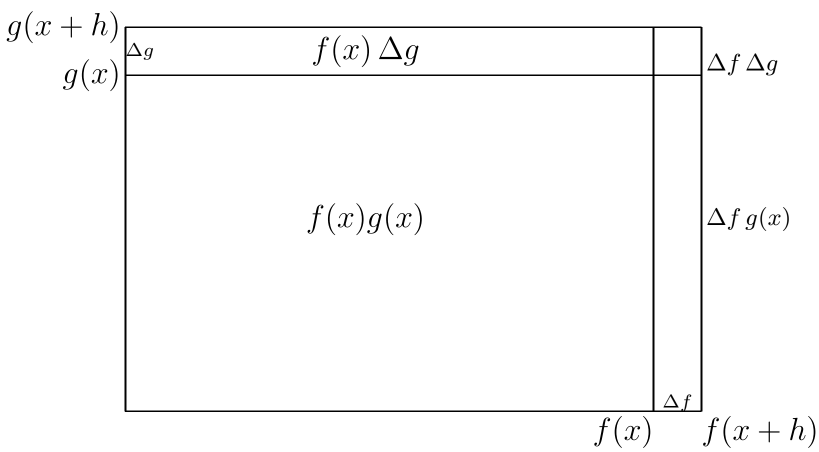
$(30 + 2) \times (50 + 3) =$

$30 \times 50 \; +3 \times 30 \; +2 \times 50 \; +2 \times 3$

$= 1500 \; +90 \; +100 \; +6$

$= 1696$

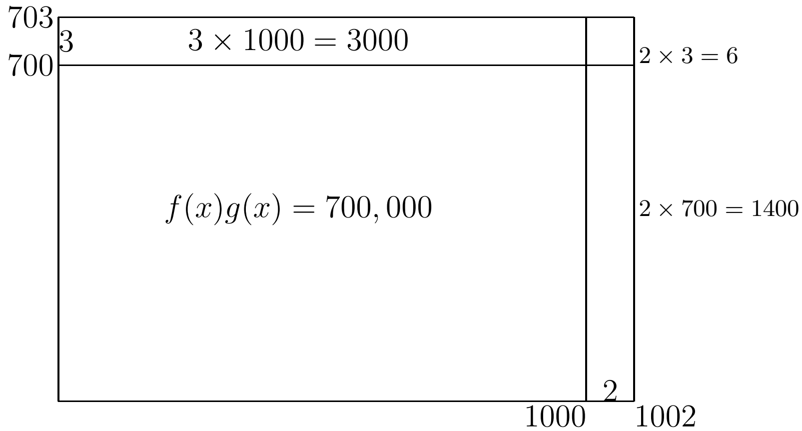
30. Picture of Product Rule



$\Delta f = f(x+h) - f(x)$
$\Delta g = g(x+h) - g(x)$

31. Numerical Example

Suppose $f(x) = 1000$ and $g(x) = 700$ and we increase $f(x)$ by $\Delta f = 2$ and $g(x)$ by $\Delta f = 3$:



The area increases by $2 \times 700 + 3 \times 1000 + 6 = 1400 + 3000 + 6 = 4406$

32. Proof of Product Rule

$$f(x+h)g(x+h) = \big[f(x) + \Delta f\big] \cdot \big[g(x) + \Delta g\big]$$
$$= f(x)g(x) + \Delta f\, g(x) + f(x)\Delta g + \Delta f \Delta g$$

$$f(x+h)g(x+h) - f(x)g(x) = \Delta f\, g(x) + f(x)\Delta g + \Delta f \Delta g$$

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{\Delta f\, g(x) + f(x)\,\Delta g + \Delta f \Delta g}{h} = \frac{\Delta f\, g(x)}{h} + \frac{f(x)\,\Delta g}{h} + \frac{\Delta f \Delta g}{h}\,\frac{h}{h} = \frac{\Delta f}{h} g(x) + f(x)\frac{\Delta g}{h} + \frac{\Delta f}{h}\frac{\Delta g}{h} \cdot h$$

33. Conclusion of Proof

- $\displaystyle \lim_{h\to 0} \frac{\Delta f}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$
- $\displaystyle \lim_{h\to 0} \frac{\Delta g}{h} = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$
- $\displaystyle \frac{d}{dx}[f(x)g(x)] \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$
  $\displaystyle = [\lim_{h\to 0} \left[\frac{\Delta f}{h} g(x) + f(x)\frac{\Delta g}{h} + \frac{\Delta f}{h}\frac{\Delta g}{h} h\right]$

$$= \left[\lim_{h\to 0} \frac{\Delta f}{h}\right] g(x) + f(x) \left[\lim_{h\to 0} \frac{\Delta g}{h}\right] + 0$$
$$= f'(x)g(x) + f(x)g'(x)$$

34. The Product Rule

$$\boxed{(f \cdot g)' = f' \cdot g + f \cdot g'}$$

In words this says: the derivative of a prdouct is the sum of the derivative of the first function times the second plus the first function times the derivative of the second.

Notice that the constant times rule is a special case of the product rule and the fact that the derivative of a constant function is zero:

$(cf)' = (c)'f + c(f') = 0 \cdot f + c \cdot f' = c \cdot f'$

35. Example

- $f(x) = x^5 - 6x^2$
- $g(x) = x^3 + 10x$
- $f'(x) = 5x^4 - 12x$
- $g'(x) = 3x^2 + 10$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- $= (5x^4 - 12x)(x^3 + 10x) + (x^5 - 6x^2)(3x^2 + 10)$
- $5x^7 + 50x^5 - 12x^4 - 120x^2\ 3x^7 + 10x^5 - 18x^4 - 60x^2$
- $= 8x^7 + 60x^5 - 30x^4 - 180x^2$

36. Mulipl First

- $f(x) = x^3 + 11x$
- $g(x) = x^5 - 6x^2$
- By the product rule we got $(fg)'(x) = 8x^7 + 60x^5 - 30x^4 - 180x^2$
- $p(x) = f(x)g(x) = (x^3 + 10x)(x^5 - 6x^2)$
- $= x^8 + 10x^6 - 6x^5 - 60x^3$
- $p'(x) = 8x^7 + 60x^5 - 30x^4 - 180x^2$
- the same answer

37. Warning

- The product rule does NOT say that the derivative of a product is the product of the derivatives.
- For example, what is wrong with the following derivation:
  $\frac{d}{dx}x^2 = \frac{d}{dx}x \cdot \frac{d}{dx}x = 1 \cdot 1 = 1$?
- Answer: You are not using the product rule correctly.
- The correct use of the product rule gives
  $\frac{d}{dx}x^2 = \left(\frac{d}{dx}x\right) \cdot x + x \cdot \left(\frac{d}{dx}x\right) = x \cdot 1 + 1 \cdot x = 2x$
- the correct answer.

38. The Quotient Rule

Suppose that $f(x)$ and $g(x)$ are two functions whose derivatives we know, and that we want to find the derivative of the quotient

$$r(x) = \frac{f(x)}{g(x)}$$

Dropping the "(x)" we have that

$$f = r \cdot g$$

Using the product rule,

$$f' = r' \cdot g + r \cdot g'$$

Now solve for r'

$$\begin{aligned} r' \cdot g &= f' - r \cdot g' \\ &= f' - \tfrac{f}{g} \cdot g' \end{aligned}$$

39. The Quotient Rule Continued

$$r' \cdot g = f' - \frac{f}{g} \cdot g'$$

Multiply through by $1/g$:

$$r' = \frac{1}{g}\left(f' - \frac{f}{g} \cdot g'\right) = \frac{f'}{g} - \frac{f\,g'}{g^2}$$

$$= \frac{f'\,g}{g^2} - \frac{f\,g'}{g^2} = \frac{f'\,g - f\,g'}{g^2}$$

This last formula is the quotient rule:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}}$$

40. Product vs Quotient Rule

Product Rule: $(f \cdot g)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$

Note that we can obtain the quotient rule from the product rule by

- changing the $+$ to a $-$ in the numerator
- dividing by g^2

Warning: It doesn't matter if you reverse the terms in the product rule, but it does matter in the quotient rule.

41. Old Problem Revisited

Compute the derivative of $y = \frac{1}{x^2}$ using the quotient rule.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(1) \cdot (x^2) - 1 \cdot \frac{d}{dx}(x^2)}{(x^2)^2} = \frac{0 \cdot (x^2) - 1 \cdot (2x)}{x^4}$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

Compare this answer with the result from using the power rule.

42. Quotient Rule Example

Compute the derivative of $y = \frac{x^2 - 2}{x^3 + 1}$

By the quotient rule,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\big(x^2-2\big)\cdot(x^3+1)-(x^2-2)\cdot\frac{d}{dx}(x^3+1)}{(x^3+1)^2} = \frac{\big(2x\big)\cdot(x^3+1)-(x^2-2)\cdot(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4+2x-(3x^4-6x^2)}{(x^3+1)^2}$$

$$= \frac{-x^4+6x^2+2x}{(x^3+1)^2}$$

Aren't you glad you don't have to use the limit definition?