Math 211 Lecture Notes: Sections 1.7 - 1.8

1. Relative Weights

In terms of pounds,

animal	average weight
elephant	9,000 pounds
bear	600 pounds
human	180 pounds

Thus an elephant weighs 15 times more than a bear

a bear weighs $3\frac{1}{3}$ times more than a human

Question: How much more does an elephant weigh than a human?

Answer: Multiply: An elephant weighs

$$15 \times \frac{10}{3} = 50$$

times as much as a human

2. Relative Speeds

In terms of mph (miles per hour),

vehicle	average speed
airplane	600 mph
car	60 mph
bicycle	15 mph
walker	3 mph

Thus a plane travels 10 times faster than a car

a car travels 4 times faster than a bike

a bike travels 5 times faster than a person walking

Question: How much faster is traveling in an airplane compared to walking?

Answer: Multiply: An airplane travels

$$10 \times 4 \times 5 = 200$$

times as fast as a human on foot.

3. The Moral

The moral to these two examples is that

Rates Multiply!

This is a famous rule of calculus, called the chain rule which says

If we have three variable x, y, and z,

If z is changing n times faster than y

and y is changing m times faster than x,

then z is changing $m \cdot n$ times faster than x.

4. The Chain Rule

Since the derivate tells us the rate of change, the fact that rates multiply can be written succintly as

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

This formula is called the Chain Rule

When measuring weights, our first example becomes

$$\frac{d\,\text{elepahnt}}{d\,\text{human}} = \frac{d\,\text{elepahnt}}{d\,\text{bear}} \cdot \frac{d\,\text{bear}}{d\,\text{human}}$$

5. The Chain Rule Continued

When measuring speed, the second example shows that you can string sevel rates togerther:

$$\frac{d\,\text{plane}}{d\,\text{pedestrian}} = \frac{d\,\text{plane}}{d\,\text{car}} \cdot \frac{d\,\text{car}}{d\,\text{bike}} \cdot \frac{d\,\text{bike}}{d\,\text{pedestrian}}$$

which illustrates why the rule is called the chain rule.

6. Using the Chain Rule

Suppose

$z = y^2$ and $y = x^3 + 11$

Use the chain rule to find $\dfrac{dz}{dx}$

$\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}$

$= 2y \cdot 3x^2 = 2(x^3 + 1) \cdot 3x^2$

7. Direct Calculation

$z = y^2$ and $y = x^3 + 11$

So $z = \left(x^3 + 11\right)^2 = x^6 + 22x^3 + 121$

So $\dfrac{dz}{dx} = 6x^5 + 66x^2$

This is equivalent to our previous solution:

$\dfrac{dz}{dx} = 2(x^3 + 11) \cdot 3x^2 = 6x^2(x^3 + 11)$

$= 6x^5 + 66x^2$

8. Prime Notation Version

Supose we combine two functions f and g to get

$$f \circ g(x) = f(g(x))$$

Then the Chain Rules becomes

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

In words this says: the derivative of a composition is the derivative of the outer function evaluated at the inner function times the derivative of the inside part.

9. Examples

Supose $f(x) = x^2$ and $g(x) = x^3 + 11$

Then $f'(x) = 2x$ and $g'(x) = 3x^2$

So $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$
$= 2g(x) \cdot 3x^2$
$= 2\big(x^3 + 11\big) \cdot 3x^2$

We have seen this example before.

10. Second Example

Differentiate $y = \sqrt{7x^3 + x^2 + 1}$

We are using two functions:
$f(x) = \sqrt{x}$ and $g(x) = 7x^3 + x^2 + 1$.

Since $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = 21x^2 + 2x$

So $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2}(7x^3 + x^2 + 1)^{-1/2} \cdot (21x^2 + 2x)$$

11. Chaining the Chain Rule

Differentiate $f(x) = \sqrt{x + \sqrt{x + 1}}$

Here we must use the chain rule twice.

The secret is to write the square roots as exponents:

Differentiate $f(x) = \big(x + (x+1)^{\frac{1}{2}}\big)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}\big(x + (x+1)^{\frac{1}{2}}\big)^{-\frac{1}{2}} \cdot \frac{d}{dx}\Big(x + (x+1)^{1/2}\Big)$$

$$f'(x) = \frac{1}{2}\big(x + (x+1)^{\frac{1}{2}}\big)^{-\frac{1}{2}} \cdot \Big(1 + \frac{d}{dx}(x+1)^{1/2}\Big)$$

$$f'(x) = \frac{1}{2}\big(x + (x+1)^{\frac{1}{2}}\big)^{-\frac{1}{2}} \cdot \Big(1 + \frac{1}{2}(x+1)^{-1/2}\Big)$$

12. The Rules Combined

Differentiate $f(x) = x\sqrt{2x+1}$

By the product rule , chain rule, and power rule,

$$f'(x) = 1 \cdot \sqrt{2x+1} + x \cdot \frac{d}{dx}(2x+1)^{1/2}$$

$$= \sqrt{2x+1} + x \cdot \frac{1}{2}(2x+1)^{-1/2} \cdot \frac{d}{dx}(2x+1)$$

$$= \sqrt{2x+1} + \frac{1}{2}x(2x+1)^{-\frac{1}{2}} \cdot 2$$

$$= \sqrt{2x+1} + x(2x+1)^{-\frac{1}{2}}$$

13. The Rules Combined II

Differentiate $f(x) = \sqrt{\frac{x}{x+1}}$

By the chain rule , power rule, and quotient rule,

$$f'(x) = \frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}} \cdot \frac{\frac{d}{dx}(x) \cdot (x+1) - x \cdot \frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}} \frac{1(x+1) - x(1)}{(x+1)^2}$$

$$= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}} \frac{1}{(x+1)^2}$$

14. The Second Derivative

Since we love differentiating so much, it seems natural to keep doing it.

That is, once we find the derivative of a function f, we may wish to find the derivative of the derivative f'

For example, if
$f(x) = 7x^3 + 3x^2 + 11x - 9$

then
$f'(x) = 21x^2 + 6x + 11$

The derivative of f' is
$(f')'(x) = 42x + 6$

- The derivative of the derivative is called the second derivative.
- What else would it be called?
- and is written $f''(x)$

15. Higher Order Derivatives

- There is no need to stop at two.
- The derivative of the seoncd derivative is called the third derivative What else would it be called?
- and is written $f'''(x)$
- continuing with derivatives of order 4, 5, 6, etcetera
- For $n > 3$ we write the nth derivative as
- $f^{(n)}(x)$
- Note the parentheses around the n

16. An Example

- For $f(x) = x^5$
- $f'(x) = 5x^4$
- $f''(x) = 5 \cdot 4x^3$
- $f'''(x) = 5 \cdot 4 \cdot 3x^2$
- $f^{(4)}(x) = 5 \cdot 4 \cdot 3 \cdot 2x$
- $f^{(5)}(x) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is written 5!
- and is called “5–factorial”

- For $n > 5$
- $f^{(n)}(x) = 0$

17. A Harder Example

- For $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$
- $f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$
- $f''(x) = -\frac{1}{2}\frac{-1}{2}(1-x)^{-\frac{3}{2}} \cdot (-1) = -\frac{1}{2}\frac{1}{2}(1-x)^{-\frac{3}{2}}$
- $f'''(x) = -\frac{1}{2}\frac{1}{2}\frac{-3}{2}(1-x)^{-\frac{5}{2}} \cdot (-1) = -\frac{1}{2}\frac{1}{2}\frac{3}{2}(1-x)^{-\frac{5}{2}}$
- $f^{(4)}(x) = -\frac{1}{2}\frac{1}{2}\frac{3}{2}\frac{-5}{2}(1-x)^{-\frac{7}{2}} \cdot (-1) = -\frac{1}{2}\frac{1}{2}\frac{3}{2}\frac{5}{2}(1-x)^{-\frac{7}{2}}$
- We could keep going
- Notice that the each derivative is always negative
- The higher order derivatives will never equal zeo

18. Leibnitz Notation

The second derivative in Leibnitz's notation is $\frac{d}{dx}\left(\frac{d}{dx}y\right)$ which is written as $\frac{d^2}{dx^2}y$ or $\frac{d^2y}{dx^2}$

In general

number	derivative
first	$\frac{dy}{dx}$
second	$\frac{d^2y}{dx^2}$
third	$\frac{d^3y}{dx^3}$
fourth	$\frac{d^4y}{dx^4}$
n–th	$\frac{d^ny}{dx^n}$

19. Velocity and Acceleration

The rate at which the position of an automobile (or other vehicle) is changing with time is called its velocity.

Velocity is measured in miles per hour.

The rate at which the velocity changes is called the acceleration.

Acceleration is measured in miles per hour per hour.

20. Free Fall

On Earth, when an object is dropped, the distance it falls after t seconds, if we ignore air resistance, is

$$s = 16t^2$$

Find the velocity and acceleration

$$v = \frac{ds}{dt} = 32t$$

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 32$$

called the acceleration due to gravity.

21. Skydiving

The distance a skydiver falls in 10 seconds is

$$s = 16 \cdot 10^2 = 1600$$

The skydiver's speed after 60 seconds is

$$v = 32 \cdot 10 = 320\frac{\text{ft}}{\text{sec}} \times \frac{60 \text{ sec}}{\text{min}} = 19200\frac{\text{ft}}{\text{min}}$$

To convert this to miles per hour it is convenient to know that 1 mile per hour equals 88 feet per minute:

$$v = 19200\frac{\text{ft}}{\text{min}}\frac{1 \text{ mph}}{88 \text{ ft per min}} = 218.2 \text{ mph}$$

Is it reasonable to ignore air resistance?