Exponential Functions

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1. Exponential Functions

An exponential functions has the form

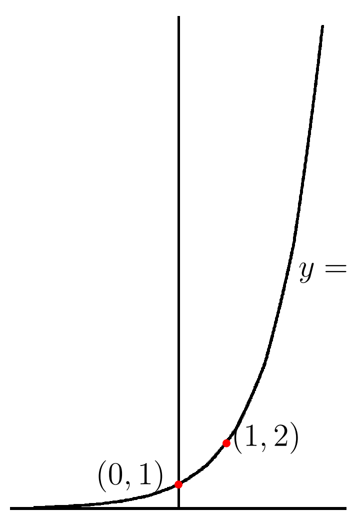
$$f(x) = a^x$$

where the base number a is a positive real, not equal to 1.

For $a = 2$ here is a table of values of a^x:

x	1	2	3	4	5	6	7	8
2^x	2	4	8	16	32	64	128	256

2. Graph of $y = 2^x$



3. Rules of exponents

(1) $a^m \cdot a^n = a^{m+n}$
(2) $\dfrac{a^m}{a^n} = a^{m-n}$
(3) $\left(a^m\right)^n = a^{mn}$
(4) $a^0 = 1$
(5) $a^{-n} = \dfrac{1}{a^n}$
(6) $a^{1/n} = \sqrt[n]{a}$

4. The Square root of a Million

To compute the square root of a million

nobody can do that it their head!

First write 1 million as 10^6

Using the rule $\sqrt{x} = x^{1/2}$

and the rule $\left(a^m\right)^n = a^{mn}$

the square root of a million is

$\left(10^6\right)^{\frac{1}{2}} = 10^{(6 \cdot \frac{1}{2})} = 10^3 = 1000$

The square root one million is one thousand.

5. The Derivative of $y = a^x$

Let $f(x) = a^x$ and $g(x) = f(x+b) = a^{x+b}$

We will compute the derivative of $g(x)$ in two ways.

First Way:

By the chain rule,

$g'(x) = f'(x+b) \cdot \frac{d}{dx}(x+b) = f'(x+b) \cdot 1 = f'(x+b)$

Second Way:

By the addition rule for exponents,

$g(x) = a^{x+b} = a^b \cdot a^x = a^b \cdot f(x)$

Since a^b is a constant as far as the variable x is concerned,

$g'(x) = a^b \cdot f'(x)$

Equating these two formula for $g'(x)$ gives us

$f'(x + b) = a^b \cdot f'(x)$

6. The Derivative of $y = a^x$

Now take

$f'(x + b) = a^b \cdot f'(x)$

and plug in $x = 0$:

$f'(0 + b) = a^b \cdot f'(0)$

or $f'(b) = c \cdot a^b$

where the constant $c = f'(0)$

Since we would like our variable to be x and not b, we just replace b by x in this formula:

$f'(x) = c \cdot a^x, \quad c = f'(0)$

7. Summary

The derivative of an exponential function is that same exponential function times a constant.

$$\boxed{\frac{d}{dx}a^x = c \cdot a^x}$$

The constant c is the derivative of a^x at $x = 0$

8. The Constant

At this point you may be wondering,

how do we know what the constant c is?

Since $c = f'(0)$ we could ise the limit definition of derivative to evaluate c:

For $f(x) = a^x$

$$c = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{a^h - a^0}{h}$$

$$= \lim_{h\to 0} \frac{a^h - 1}{h}$$

For a given value of a we could evaluate this limit, by taking values of h closer and closer to zero and examining the results.

9. The Constant when $a = 2$

x	$\frac{2^h - 1}{h}$
.1	.7177346
.01	.6955555
.001	.6933875
.0001	.6931712
.00001	.6931496
.000001	.6931474
.0000001	.6931472

It looks like the constant when $a = 2$ is $c = .6931470$

So $\frac{d}{dx}2^x = 0.6931472 \cdot 2^x$

10. Constant c for $a = 2, \ldots, 10$

$f(x)$	$c = \lim_{h\to 0} \frac{a^h - 1}{h}$
2^x	.6931472
3^x	1.0986122
4^x	1.3862944
5^x	1.6094379
6^x	1.7917595
7^x	1.9459101
8^x	2.0794415
9^x	2.1972245
10^x	2.3025851

Observe that for some value of a between 2 and 3, this constant will be exactly 1.

11. The function e^x

In calculus, the most commonly used base, by far, is the one for which the constant $c = 1$.

This base is so special, it has its own single-letter name: e

It was named in honor of the Swiss mathematician who disovered this number: Leonard Euler

The value of Euler's constant to twentyfive digits is

$$e = 2.718281828459045235360287$$

The function e^x is on on your scientific or graphing calculator.

12. What's important about e^x

The single most important fact about e^x is that

since $c = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$$\boxed{\text{The derivative of } y = e^x \text{ is } y' = e^x}$$

in other words, e^x is its own derivative.

What's the second derivative of e^x?

What's the third derivative of e^x?

What's the hundredth derivative of e^x?

13. Working with e^x

In computing derivatives involving the exponential function e^x, just use the chain rule, product rule, and quotient rule.

Here are some examples:

14. Example 1.

Differentiate $y = e^{3x+1}$

By the chain rule,

$$\begin{aligned} y' &= e^{3x+1} \cdot \frac{d}{dx}(3x+1) \\ &= e^{3x+1} \cdot 3 \end{aligned}$$

which is usually written $3e^{3x+1}$

15. Example 2.

Differentiate $y = e^{x^2-2x+3}$

By the chain rule,

$$\begin{aligned} y' &= e^{x^2-2x+3} \cdot \frac{d}{dx}(x^2-2x+3) \\ &= e^{x^2-2x+3}(2x-2) \end{aligned}$$

16. Example 3.

Differentiate $y = x^2 e^x$

By the product rule,

$$y' = \frac{d}{dx}\big(x^2\big)e^x + x^2\frac{d}{dx}(e^x)$$
$$= 2xe^x + x^2e^x$$

17. Tangent Line

Find the tangent line to $y = f(x) = e^{2x-4}$ at $(2, 1)$

Differentiate: $f'(x) = 2e^{2x-4}$

The slope of the tangent line is

$m = f'(2) = 2e^{2\cdot 2-4} = 2e^0 = 2$

By the point–slope formula, the tangent line is

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$y = 2x - 3$