

Integration

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1. TWO FUNDAMENTAL PROBLEMS OF CALCULUS

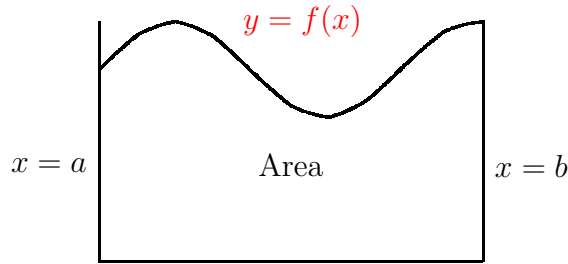
- First Problem: Study how one variable changes with respect to another
 - How does area change with respect to radius
 - How does temperature change with respect to altitude
 - How does y change with respect to x
- Second Problem: Find the area under a curve
- We use **derivatives** to solve Problem 1
- We use **integrals** to solve Problem 2

2. THE AREA PROBLEM

Given a curve $y = f(x)$

For now we will assume that $f(x)$ is ≥ 0

How do we find the area under the curve and above the x -axis, between two vertical sides $x = a$ and $x = b$?



3. A CASE WE CAN DO

If the curve is linear, then we can use the formula for the area of a triangle

$$A = \frac{1}{2} \text{ base} \times \text{height}$$

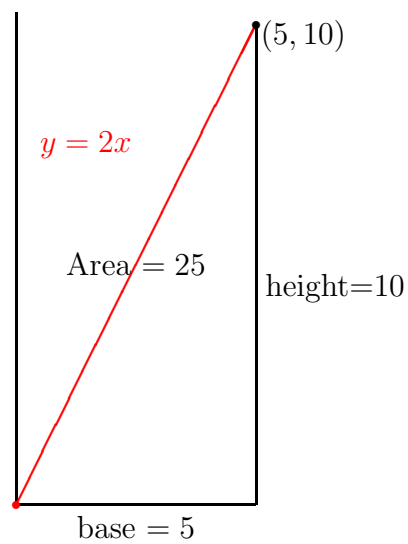
Example. $y = f(x) = 2x$

between $a = 0$ and $b = 5$

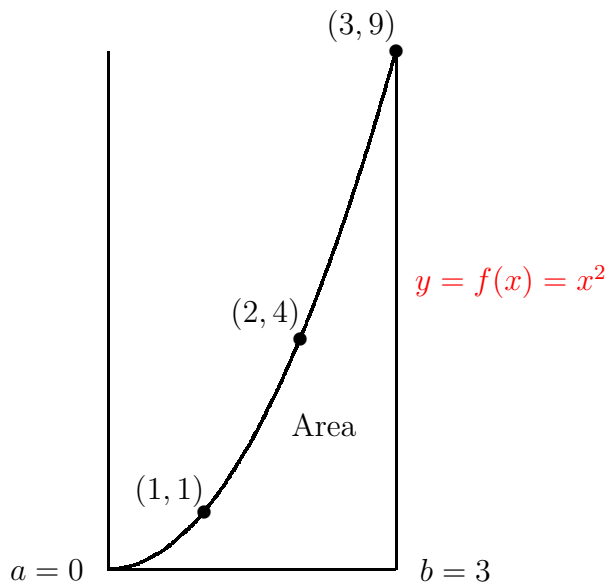
base = 5

height = 10

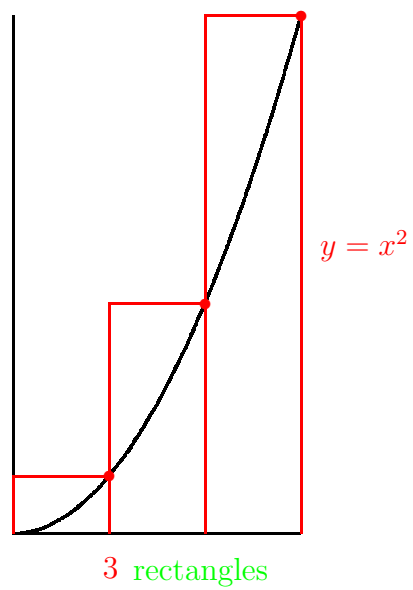
$$A = \frac{1}{2} 5 \cdot 10 = 25$$

4. GRAPH OF $y = 2x$ 

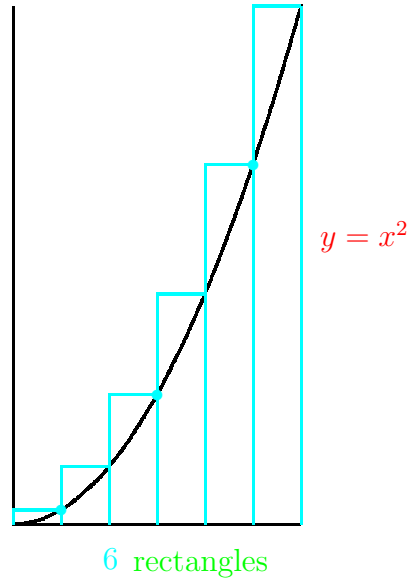
5. A CASE WE CAN'T DO



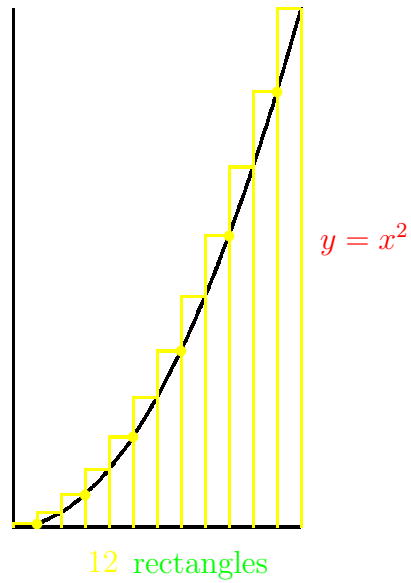
6. RECTANGULAR APPROXIMATION



7. RECTANGULAR APPROXIMATION



8. RECTANGULAR APPROXIMATION

9. SUB-DIVIDING $[a, b]$

Set-up: divide the interval $[a, b]$ into n equally spaced segments, whose division points are x_0, x_1, \dots, x_n .



The width of each segment is

$$\boxed{h = \frac{b - a}{n}}$$

The division points x_k are $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $x_3 = a + 3h$ and, in general,

$$\boxed{x_k = a + kh}$$

10. SPECIAL CASE: $a = 0$

When $a = 0$,



these formulas become

$$\boxed{h = \frac{b}{n}} \quad \text{and} \quad \boxed{x_k = kh}$$

Note that $x_n = nh = n \frac{b}{n} = b$, as it should.

11. RIEMANN METHOD

In the Riemann method, we use n rectangles to approximate the area under the curve $y = f(x)$

Each rectangle has as its base the subinterval $[x_{k-1}, x_k]$ of width h

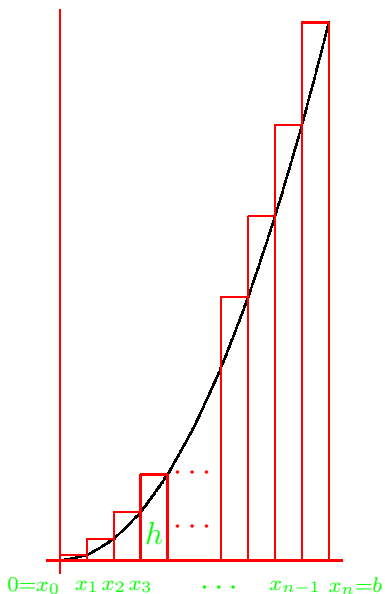
The height of each rectangle is $f(x_k)$

Since the area of a rectangle is base \times height, the k -th rectangle has area

$$h \cdot f(x_k)$$

Our approximation, called a **Riemann Sum** is the sum of the areas of these n rectangles

12. RIEMANN METHOD PICTURE



13. USING 3 SUBDIVISIONS

For the function $f(x) = x^2$, and interval $[a, b] = [0, 3]$ suppose we use $n = 3$ rectangles

We divide $[0, 3]$ into 3 equally spaced subintervals, each of width $h = 3/3 = 1$

with division points $x_k = kh = k$

$$0=x_0 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \quad x_1=1 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \quad x_2=2 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \quad x_3=3$$

Notice that the height of the k th rectangle with base $[x_{k-1}, x_k]$ is $f(x_k) = x_k^2$

k	1	2	3
x_k	1	2	3
$f(x_k)$	1	4	9

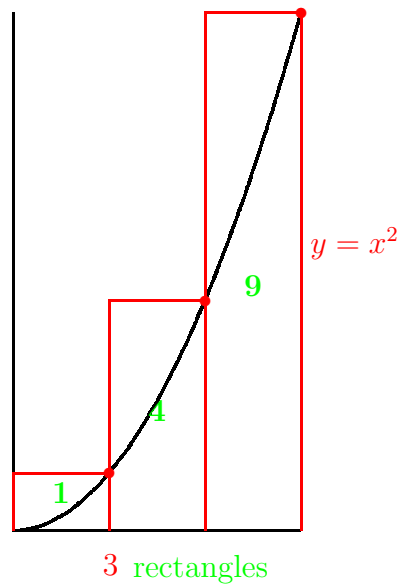
14. 3 RECTANGLES

Each rectangle has width $h = 1$ and height $f(x_k) = x_k^2$

The area of the 3 red rectangles is

$$\begin{aligned}
 A(3) &= h \cdot f(x_1) + h \cdot f(x_2) + h \cdot f(x_3) \\
 &= h \cdot x_1^2 + h \cdot x_2^2 + h \cdot x_3^2 \\
 &= 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 \\
 &= 1 + 4 + 9 \\
 &= 14
 \end{aligned}$$

15. RECTANGULAR APPROXIMATION



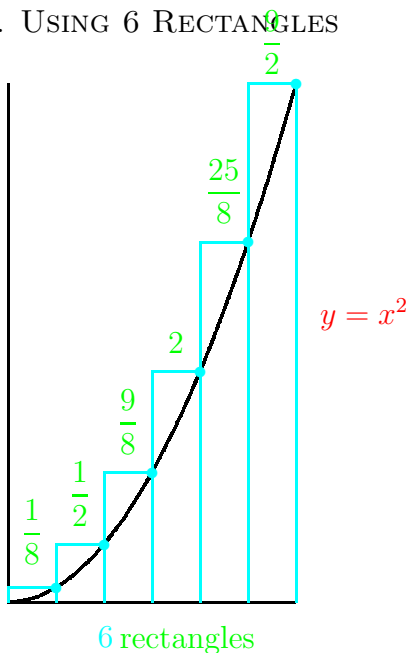
16. 6 RECTANGLES

Each rectangle has width $h = \frac{1}{2}$, subdivision point $x_k = kh = \frac{1}{2}k$, and height $f(x_k) = x_k^2$

The area of the 6 bluish green rectangles is

$$\begin{aligned}
 A(6) &= h \cdot f(x_1) + h \cdot f(x_2) + h \cdot f(x_3) + h \cdot f(x_4) + h \cdot f(x_5) + h \cdot f(x_6) \\
 &= h \cdot x_1^2 + h \cdot x_2^2 + h \cdot x_3^2 + h \cdot x_4^2 + h \cdot x_5^2 + h \cdot x_6^2 \\
 &= h \cdot \left(\frac{1}{2}\right)^2 + h \cdot (1)^2 + h \cdot \left(\frac{3}{2}\right)^2 + h \cdot (2)^2 + h \cdot \left(\frac{5}{2}\right)^2 + h \cdot (3)^2 \\
 &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot \frac{25}{4} + \frac{1}{2} \cdot 9 \\
 &= \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} = \frac{35}{8} + 7 = 11\frac{3}{8}
 \end{aligned}$$

17. USING 6 RECTANGLES



18. NOTATION TIME OUT

It is getting very tedious writing these sums

The sum with 12 rectangles will be even worse!

To ease the writing, mathematicians invented the summation symbol \sum

It works like this:

- underneath \sum we write the starting index
- above \sum we write the ending index
- to the right of \sum we write the form of the things we are adding

19. SO WHAT?

$\sum_{k=1}^3 a_k$ is the sum $a_1 + a_2 + a_3$

It doesn't seem so bad to write these out!

$\sum_{k=1}^{50} a_k$ is the sum

$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} + a_{25} + a_{26} + a_{27} + a_{28} + a_{29} + a_{30} + a_{31} + a_{32} + a_{33} + a_{34} + a_{35} + a_{36} + a_{37} + a_{38} + a_{39} + a_{40} + a_{41} + a_{42} + a_{43} + a_{44} + a_{45} + a_{46} + a_{47} + a_{48} + a_{49} + a_{50}$

Nobody wants to write all 50 terms.

Though, in fairness we could have written this last sum as $a_1 + a_2 + a_3 + \dots + a_{49} + a_{50}$

20. SUMMATION EXAMPLES

The sum of the first 10 integers can be written

$$S = \sum_{k=1}^{10} k$$

Evaluate this sum

$$S = \sum_1^{10} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

21. SUMMATION EXAMPLES

What is $S = \sum_1^{10} 2k$

This is the sum of the first 10 even integers:

$$S = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 = 110$$

Can you think of an easier way, using our previous calculation $S = \sum_1^{10} k = 55$?

Factor out the 2 from each term

$$S = \sum_1^{10} 2k = 2 \sum_1^{10} k = 2 \cdot 55 = 110$$

22. RECTANGULAR APPROX OF AREA

Using summation notation, we can write the sum of the areas of the 6 rectangles used to approximate the area under the curve

$$A(6) = h \cdot f(x_1) + h \cdot f(x_2) + h \cdot f(x_3) + h \cdot f(x_4) + h \cdot f(x_5) + h \cdot f(x_6)$$

$$\text{as } A(6) = \sum_1^6 h f(x_k)$$

To go from 6 to 12 rectangles is easy:

$$A(12) = \sum_1^{12} h f(x_k) = h \sum_1^{12} f(x_k)$$

by factoring out h .

23. EXAMPLE $y = x^2$ REVISITED

$$y = f(x) = x^2$$

$$[a, b] = [0, 3]$$

$$h = \frac{3}{n}$$

$$x_k = hk$$

$$A(n) = h \sum_{k=1}^n f(x_k)$$

24. COMPUTING THE EXACT AREA

The following table gives the values of $A(n)$ for various values of n :

n	$A(n)$
3	14
6	11.375
12	10.15625
24	9.5703125
100	9.13545
1000	9.0135045
10000	9.001350045
1000000	9.0000135

It looks like the exact area under the curve $y = x^2$ between 0 and 3 is 9

25. CALCULATING AREAS

The exact area A under the curve $y = f(x)$ from a to b is defined to be the limit of the Riemann Sums $A(n)$ as the number of rectangles n goes to infinity.

That is,

$$A = \lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} h \sum_{k=1}^n f(x_k)$$

For many simple functions, such as $f(x) = x^n$ the limit that gives the area under the curve has been worked out.

26. ILLUMINATING EXAMPLE

For example, suppose $f(x) = x^2$ and we wish to compute the area from $a = 0$ to $b = x$.

It can be shown that

$$A(n) = \frac{x^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$$

so that

$$A = \lim_{n \rightarrow \infty} A(n) = \lim_{n \rightarrow \infty} \frac{x^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = \frac{x^3}{6} (1 + 0) (2 + 0) = \frac{x^3}{3}$$

When $x = 3$, $A = \frac{3^3}{3} = 9$, a fact that we laboriously worked out earlier.

27. AREA UNDER $y = x^n$

Suppose $f(x) = x^n$ for any non-negative integer.

We want to find the area A under the curve from $a = 0$ to $b = x$.

Case: $n = 0$

Our function is just $y = x^0 = 1$.

The region is one long rectangle, whose dimensions are x by 1.

The area of this rectangle is $A = x$.

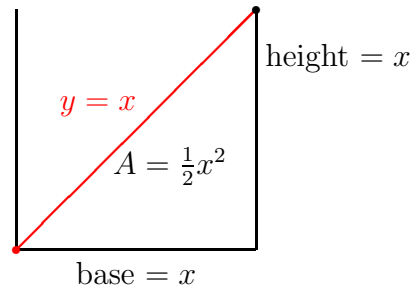
28. AREA UNDER $y = x^1$

Case: $n = 1$

Our function is just $y = x^1 = x$.

The region is an isosceles triangle whose dimensions are x by x .

The area of this rectangle is $A = \frac{1}{2}x^2$.



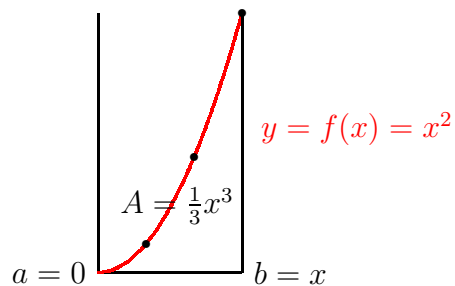
29. AREA UNDER $y = x^2$

Case: $n = 2$

Our function is just $y = x^2$.

We have already worked this out.

The area of this region is $A = \frac{1}{3}x^3$.



30. AREA UNDER $y = x^n$, $n = 0, \dots, 6$

For every possible value of the n , the area under the curve $y = f(x) = x^n$ from 0 to x has been calculated.

These areas are given in the following chart:

$f(x)$	A
$x^0 = 1$	x
x^1	$\frac{1}{2}x^2$
x^2	$\frac{1}{3}x^3$
x^3	$\frac{1}{4}x^4$
x^4	$\frac{1}{5}x^5$
x^5	$\frac{1}{6}x^6$
x^6	$\frac{1}{7}x^7$

Differentiate the second column and compare with the first column.

31. BIG SURPRISE!

It looks like

the derivative of the area is just the function $f(x)$

That is, if $A(x)$ = the area under the curve $y = f(x)$ from a to x , then

$$A'(x) = f(x)$$

So in order to solve the area problem, we must find a function whose derivative is $f(x)$

Such functions are called **anti-derivatives**

32. ANTI-DERIVATIVES

We say that $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Notice that the letters are case-sensitive.

Essentially, you worked out the derivative of a function $F(x)$, got the answer $f(x)$.

Can you figure out what was the original function just from knowing its derivative?

The answer is “sometimes”

Suppose you know

$$F'(x) = 3x^2 - 14x + 11$$

What's $F(x)$?

$$F(x) = x^3 - 7x^2 + 11x \quad \text{works}$$

$$\text{But so does } F(x) = x^3 - 7x^2 + 11x + 17$$

33. THE TROUBLE WITH CONSTANTS

Since the derivative of any constant is zero, you cannot know the value of that constant just from the derivative.

The following fact shows that, except for an undetermined constant, antiderivatives are essentially unique:

If two functions $f(x)$ and $g(x)$ have the same derivative, then $f(x) = g(x) + C$, for some constant C .

When asked to find the antiderivative of

$$F'(x) = 3x^2 - 14x + 11$$

the most general answer is to write

$$F(x) = x^3 - 7x^2 + 11x + C$$

where C is a constant.

34. INTEGRAL NOTATION

Mathematicians write the symbol

$$\int f(x) dx$$

to mean the anti-derivative of $f(x)$

The symbol \int is called the “integral sign” and the anti-derivative of $f(x)$ is also called the **integral** of $f(x)$.

Thus,

$$\int 3x^2 - 14x + 11 dx = x^3 - 7x^2 + 11x + C$$

35. THE POWER RULE

For any value of n except -1 ,

$$\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1} + C}$$

You can easily check this formula by differentiation:

$$\begin{aligned} & \frac{d}{dx} \frac{1}{n+1}x^{n+1} + C \\ &= \frac{1}{n+1}(n+1)x^{(n+1)-1} \\ &= x^n \end{aligned}$$

More good news: this formula works for rational numbers, negative numbers, and real numbers, as well as integers.

36. EXAMPLES

$$\begin{aligned} \int x^{10} dx &= \frac{1}{11}x^{11} + C \\ \int \frac{1}{x^3} dx &= \int x^{-3} dx = \frac{1}{-2}x^{-2} + C = -\frac{1}{2}x^{-2} + C \\ \int \sqrt{x} dx &= \int x^{1/2} dx = \frac{1}{3/2}x^{3/2} + C = \frac{2}{3}x^{3/2} + C \end{aligned}$$

37. TWO RULES

We can integrate sums such as $x^3 + x$ by integrating each term separately:

$$\int x^3 + x dx = \int x^3 dx + \int x dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$

Why don't we write $C + C$?

The rule that justifies this is

$$\boxed{\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx}$$

There is also a rule for “pulling out constants”:

$$\boxed{\int a \cdot f(x) dx = a \int f(x) dx}$$

where a is a constant

38. MORE PRACTICE

$$\int 15x^2 + 2 dx = \frac{15}{3}x^3 + 2x + C$$

$$\int 6 - \frac{4}{x^2} dx = \int 6 - 4x^{-2} dx = 6x - \frac{4}{-1}x^{-1} + C = 6x + \frac{4}{x} + C$$

$$\int 6\sqrt[3]{x} dx = \int 6x^{1/3} dx = \frac{6}{4/3}x^{4/3} + C = \frac{9}{2}x^{4/3} + C$$

$$\begin{aligned} \int 10x + \frac{2}{\sqrt{x}} + 1 dx &= \int 10x + 2x^{-1/2} + 1 dx = \frac{10}{2}x^2 + \frac{2}{1/2}x^{1/2} + x + C \\ &= 5x^2 + 4x^{1/2} + x + C \end{aligned}$$

39. A QUOTIENT

Integrate $\int \frac{x^2 - x}{x} dx$

We first simplify the the fraction

$$\int \frac{x^2 - x}{x} dx = \int \frac{x(x - 1)}{x} dx = \int x - 1 dx = \frac{1}{2}x^2 - x + C$$

Question: Could we have integrated this quotient by some kind of “quotient rule” for integrals?

40. THE BAD NEWS ABOUT INTEGRATION

There is no **product rule**, **quotient rule**, or **chain rule** for integrals.

Integrate $\int (x^3 - 1)^2 dx$

There is no “chain rule” for integrals, so we have to square out the binomial:

$$\int (x^3 - 1)^2 dx = \int x^6 - 2x^3 + 1 dx = \frac{1}{7}x^7 - \frac{2}{4}x^4 + x + C$$