# Integration Continued

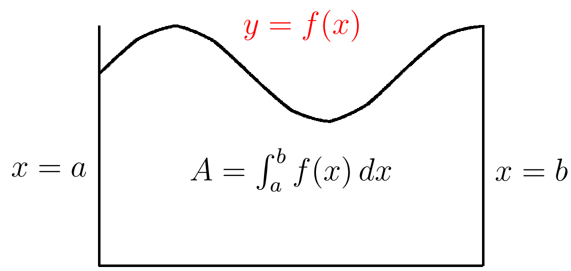
Professor Richard Blecksmith
richard@math.niu.edu
Dept. of Mathematical Sciences
Northern Illinois University
http://math.niu.edu/~richard/Math211

## 1. The Definite Integral

We define the definite integral of a continuous function $f(x)$ from $a$ to $b$, written

$$\int_a^b f(x)\,dx$$

to be the area under the curve $y = f(x)$ between two vertical sides $x = a$ and $x = b$.




## 2. Connection to anti-derivatives

Let $A(x) = \int_a^x f(t)\,dt$, the area under the curve $f(t)$ starting at $a$ and ending at $x$.

Note that we have written the function as $f(t)$ rather than as $f(x)$, because we are using $x$ as the stopping value.

Now let $F(x)$ be any anti-derivative of $f(x)$.

That is, $F(x)$ has the property

$F'(x) = f(x)$

We have seen that

$A'(x) = f(x)$

So $F(x)$ and $A(x)$ both have the same derivative $f(x)$.

But functions with the same derivative differ by a constant.

So $A(x) = F(x) + C$ for some constant $C$

## 3. Connection to anti-derivatives II

If $F(x)$ is an anti-derivative of $f(x)$, then

$A(x) = \int_a^x f(t)\,dt = F(x) + C$

To determine $C$, plug in $x = a$ into the equation

$A(a) = \int_a^a f(t)\,dt = F(a) + C$

Clearly $A(a) = \int_a^a f(t)\,dt = 0$ Why? because our interval $[a, b]$ is just the single point $a$.

So $F(a) + C = 0$

or $C = -F(a)$

and hence $A(x) = \int_a^x f(t)\,dt = F(x) - F(a)$

Switching letters $b$ for $x$ and $x$ for $t$

$\int_a^b f(x)\,dx = F(b) - F(a)$

## 4. The Fundamental Theorem of Calculus

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where $F(x)$ is an anti-derivative of $f(x)$

Calculus was born when Sir Issac Newton the fig guy? and Gottfried Wilhelm Leibniz independently disovered this essential connection between integrals (areas) and derivatives (rates of change).

The Fundamental Theorem of Calculus says that in order to find the area $A$ under the curve $y = f(x)$ from $a$ to $b$,

first find an anti-derivative of $f(x)$, that is, a function $F(x)$ whose derivative is $f(x)$

then the area is $A = F(b) - F(a)$

## 5. Easy Example

Compute the definite integral $\int_1^3 2x\,dx$

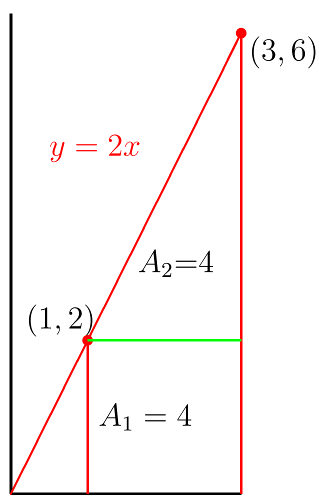
An anti-derivative of $f(x) = 2x$ is $F(x) = x^2$

So by the Fundamental Theorem of Calculus

$$\int_1^3 2x\,dx = F(3) - F(1) = 3^2 - 1^2 = 8$$

The area under the line $y = 2x$ for 1 to 3 is 8, as the next picture shows.

## 6. Picture $\int_1^3 2x = 8$




$A = A_1 + A_2 = 8$

## 7. Some Notation

Instead of writing $F(b) - F(a)$, we use notation that allows us not to refer to $F(x)$ by name

$$F(x)\Big|_a^b = F(b) - F(a)$$

Evaluate the definite integral $\displaystyle\int_2^5 3x^2\,dx$

An anti-derivative of $f(x) = 3x^2$ is $F(x) = x^3$

So by the Fundamental Theorem of Calculus

$\displaystyle\int_2^5 3x^2\,dx = x^3\Big|_2^5 = 5^3 - 2^3 = 125 - 8 = 117$

## 8. Definite vs Indefinite Integral

In evaluating the definite integral $\displaystyle\int_a^b f(x)\,dx$,

the anti-derivative $F(x)$ is just the indefinite integral $F(x) = \displaystyle\int f(x)\,dx$.

In this case, we do not need to write the $+C$ at the end of the integral Why not? because it will end up being cancelled.

For example,

$\displaystyle\int_1^3 2x\,dx = \left(x^2+C\right)\Big|_1^3 = (3^2+C)-(1^2+C) = (9+C)-(1+C) = 9-1+C-C$
$= 8$

It is WRONG!!! to write $\displaystyle\int_1^3 2x\,dx = 8 + C$

## 9. Area Problem 1

Find the area under the curve $y = \sqrt{x}$ from 1 to 9

Solution: By the Fundamental Theorem of Calculus, this area is

$$\int_1^9 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^9 = \frac{2}{3}\left[\sqrt{9}^3 - \sqrt{1}^3\right] = \frac{2}{3}(27-1) = \frac{52}{3}$$

## 10. Area Problem 2

Find the area under the curve $y = \dfrac{1}{x}$ from 2 to 7

Solution: In order to use the Fundamental Theorem of Calculus, we need an anti-derivative of $\frac{1}{x}$.

Happily we know a function whose derivative is $\frac{1}{x}$,

namely the natural log.

That is,

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

$$\int_2^7 \frac{1}{x}\,dx = \ln(x)\Big|_2^7 = \ln(7) - \ln(2) = 1.945910149 - 0.6931471806 = 1.252762968$$

## 11. Area Problem 3

Find the area under the curve $y = 2x - 10$ from 3 to 6

$$\int_3^6 2x - 10\,dx = x^2 - 10x\Big|_3^6$$

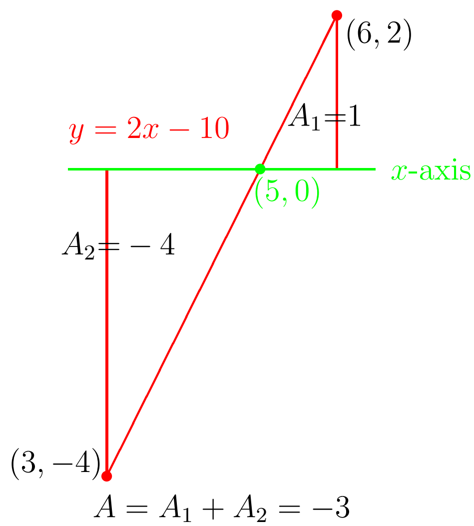
$= 36 - 60 - (9 - 30) = -24 + 21 = -3$

Wait a minute.

How can area be negative?

The answer is that integrals compute the areas of regions below the $x$-axis as negative.

## 12. Picture $\int_3^6 2x - 10\,dx = -3$




## 13. Area between two curves

The area between two curves from $a$ to $b$ is given by the integral

$$A = \int_a^b f(x) - g(x)\,dx$$

provided $f(x) \geq g(x)$ on the interval $[a, b]$

That is, to find the area between two curves, we always subtract the curve on the bottom from the one on the top.

## 14. Example

Find the area $y = f(x) = x^2 - 6x + 11$ and $y = g(x) = -x + 7$.

To find where these curve cross, we set $f(x) = g(x)$

$x^2 - 6x + 11 = -x + 7$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ or $x = 4$

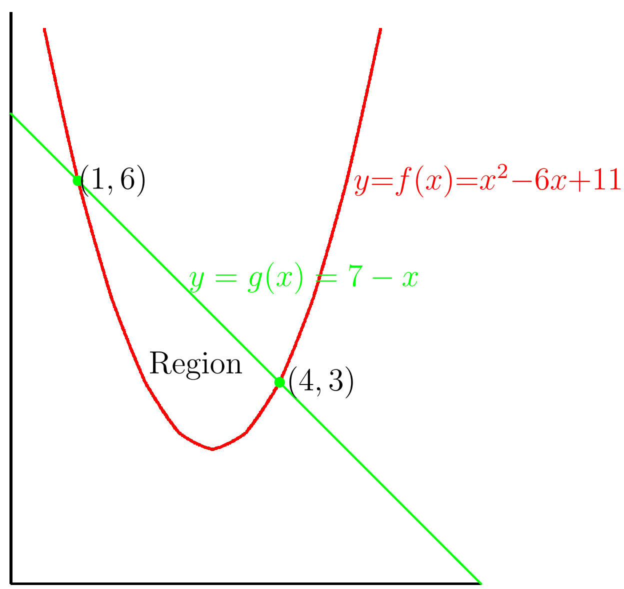
When $x = 1$, $y = f(1) = g(1) = 6$

When $x = 4$, $y = f(4) = g(4) = 3$

So the intersection points are $(1, 6)$ and $(4, 3)$

Now draw the graph to determine which curve is on top.

## 15. Tale of Two Graphs




## 16. Area Between 2 Curves

We see from the graph that $g(x) = 7 - x$ lies above $f(x) = x^2 - 6x + 11$

The area between these two curves is

$\int_1^4 g(x) - f(x)\,dx = \int_1^4 (7 - x) - (x^2 - 6x + 11)\,dx$

$= \int_1^4 7 - x - x^2 + 6x - 11\,dx$

$= \int_1^4 -4 + 5x - x^2\,dx$

$$= \left(-4x + \tfrac{5}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_1^4$$

$$= \left(-16 + \tfrac{5}{2}\,16 - \tfrac{1}{3}\,64\right) - \left(-4 + \tfrac{5}{2} - \tfrac{1}{3}\right)$$

$$= -16 + 40 - \tfrac{64}{3} + 4 - \tfrac{5}{2} + \tfrac{1}{3}$$

$$= 28 - \tfrac{63}{3} - \tfrac{5}{2}$$

$$= 7 - \tfrac{5}{2}$$

$$= \tfrac{9}{2}$$