Logarithms

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1. Definition of Logarithm

For $a > 0$, $a \neq 1$, the logarithm function

$y = \log_a(x)$ means $x = a^y$

a is called the base of the logarithm

That is, the logarithm function log_a is the inverse of the exponential function a^x.

Just like the square root function is the inverse of the squaring function.

2. Examples

$3^2 = 9$ becomes $\log_3(9) = 2$

$10^6 = 1000000$ becomes $\log_{10}(1000000) = 6$

$\log_6 \dfrac{1}{36} = -2$ because $6^{-2} = \frac{1}{36}$

$\log_2(x) = 6$ becomes $2^6 = x$

$4^{3/2} = 8$ becomes $\log_4(8) = \dfrac{3}{2}$

$y = \log_e(x)$ becomes $x = e^y$

3. Rules of logarithms

Algebraic Rules:

(1) $\log_a(mn) = \log_a(m) + \log_a(n)$
(2) $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
(3) $\log_a(m^k) = k\log_a(m)$
(4) $\log_a(1) = 0$ and $\ln(e) = 1$

Inverse Rules:

(5) $\log_a(a^x) = x$
(6) $a^{\log_a(x)} = x$

Change of base rule: 7. $\log_a b \log_b c = \log_a c$

4. Examples

$\log_{10} 6x = \log_{10} 6 + \log_{10} x$

$\log_{10} \frac{x}{100} = \log_{10} x - \log_{10} 100 = \log_{10} x - 2$

$\log_2(7^5) = 5\log_2 7$

$\log_{10} \sqrt{x} = \log_{10} x^{1/2} = \frac{1}{2}\log_{10} x$

5. The natural logarithm

In calculus, the most commonly used base, by far, for logarithms is e.

The natural logarithm—written ln—is the logarithm to the base e:

$$\boxed{\ln(x) = \log_e(x)}$$

The function ln is a key on your scientific or graphing calculator.

6. Rules of natural log

Algebraic Rules:

(1) $\ln(mn) = \ln(m) + \ln(n)$
(2) $\ln_a\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$
(3) $\ln(m^k) = k\ln(m)$
(4) $\ln(1) = 0$

Inverse Rules:

(5) $\ln(e^x) = x$
(6) $e^{\ln(x)} = x$

Change of base rule: 7. $\log_a(x) = \frac{1}{\ln a}\ln(x)$

7. Rules of natural log

The change of base rule

$$\log_a(x) = \frac{1}{\ln a}\ln(x)$$

allows your calculator to compute logarithms to any base:

Example:

$\log_2(7) = \ln 7/\ln 2 = 1.945910149/.6931471806 = 1.348802133$

8. Derivative of $\ln x$

Let $y = f(x) = \ln x$

Rewrite $y = \ln x$ as an exponential function

$e^y = x$ or $e^{f(x)} = x$

Now differentiate both sides of this equation with respect to x:

$\frac{d}{dx}e^{f(x)} = \frac{d}{dx}x$

$e^{f(x)}f'(x) = 1$

So $f'(x) = \dfrac{1}{e^{f(x)}} = \dfrac{1}{x}$

9. What's important about $\ln x$

The single most important fact about $\ln x$ is that

$$\boxed{\text{The derivative of } y = \ln x \text{ is } y' = \frac{1}{x}}$$

There is no power function $y = x^n$ that has this property.

For if we try to find a value of n such that

$\dfrac{d}{dx}x^n = nx^{n-1} = x^{-1}$

we see that $n - 1 = -1$ or n must be 0

But the function $y = x^0$ is just the constant function $y = 1$

and clearly the derivative of $y = 1$ is not $1/x$.

10. Behavior of $\ln x$

Domain of ln = Range of e^x: $(0, \infty)$

Range of ln = Domain of e^x: $(-\infty, \infty)$

$y = \ln x$ if and only if $x = e^y$

The graph of $y = \ln x$ has the same shape as the graph of $y = e^x$, with the x and y reversed.

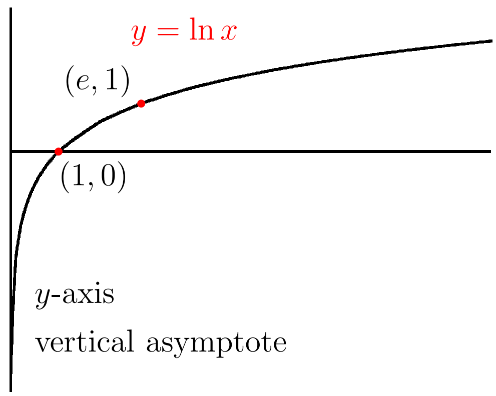
The graph of $\ln x$ is always increasing because $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ is positive (since $x > 0$)

We obtain the convavity of the graph of $\ln x$ from the second derivative:

$\dfrac{d^2}{dx^2}\ln x = \dfrac{d}{dx}\dfrac{1}{x} = -\dfrac{1}{x^2} < 0$

A negative second derivative implies the graph is concave down.

11. Graph of $\ln x$



12. Working with $\ln x$

In computing derivatives involving the natural log function $\ln x$, just use the chain rule, product rule, and quotient rule.

Sometimes it helps (a lot) to use the laws of logarithms to simplify before you differentiate.

Here are some examples:

13. Example 1.

Differentiate $y = \dfrac{x^2}{\ln(x)}$

By the quotient rule,

$$y' = \frac{2x \cdot \ln(x) - x^2 \cdot \frac{1}{x}}{(\ln x)^2}$$

$$= \frac{2x \cdot \ln(x) - x}{(\ln x)^2}$$

Warning: There is no rule of logarithms for simplifying $(\ln x)^2$ It is NOT the same as $2 \ln x$

$\ln(x^2) = 2\ln(x)$ but

$(\ln x)^2$ does not simplify

14. Example 2.

Differentiate $y = \ln\left(\frac{x+1}{x-1}\right)$

Although you could use the chain rule and the quotient rule directly, it is much easier to simplify first, using the rule

$\ln_a\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$

$y = \ln(x+1) - \ln(x-1)$

$y' = \frac{1}{x+1} - \frac{1}{x-1}$

15. Example 3.

Differentiate $y = \ln\sqrt[3]{x+1}$

Simplify first using the rule

$\ln(m^k) = k\ln(m)$

$y = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$

$y' = \frac{1}{3}\frac{1}{x+1} = \frac{1}{3(x+1)}$

16. Find the function

Tell me a function $f(x)$ whose derivative satisfies

$$f'(x) = f(x)$$

Answer: $f(x) = e^x$

Can you think of any others?

Hint: Try $f(x) = 7e^x$

Then $f'(x) = 7e^x = f(x)$

7 could be any constant, so the general solution is

$f(x) = c\,e^x$

where c is a constant.

17. Find the function

Tell me a function $f(x)$ whose derivative satisfies

$$f'(x) = 2f(x)$$

Answer: $f(x) = e^{2x}$

because

$f'(x) = e^{2x} \cdot 2 = 2f(x)$

Can you think of any others?

$f(x) = c\,e^{2x}$

where c is a constant.

18. Summary

If k is a constant, the function $f(x)$ satisfies

$$f'(x) = kf(x)$$

if and only if $f(x) = c\,e^{kx}$

Note that the constant c can be determined by plugging in $x = 0$

$f(0) = c\,e^0 = c \cdot 1 = c$

that is, $c = f(0)$

The equation $f'(x) = kf(x)$ is an example of a differential equation which simply means an equation that contains derivatives.

19. Exponential Growth Problem

A freshly inoculated bacterial culture of streptococcus (a common group of micro-organisms that cause strep throat) contains 100 cells. When the culture is checked 60 minutes later, it is determined there are 450 cells present.

Assuming exponential growth, find an equation for the number of cells present at time t (measured in minutes).

Find the "doubling" time, that is, the time it takes the bacteria to double in size.

20. Exponential Growth Setup

Let $P(t)$ be the population at time t.

Exponential growth means that

$$P'(t) = kP(t)$$

where k is the population-growth constant.

The solution is

$P(t) = c\,e^{kt}$

where $c = P(0)$

Since we are told that the initial population is 100 cells, we know

$c = P(0) = 100$

So our equation is

$$P(t) = 100e^{kt}$$

The problem is to determine k.

21. Finding k

$$P(t) = 100e^{kt}$$

We know an additional piece of information:

When $t = 60$ minutes, the population is 450

that is, $P(60) = 450$

Plugging these values into the equation gives

$P(60) = 100e^{k60} = 450 \quad \text{or}$

$100e^{60k} = 450 \quad \text{or}$

$e^{60k} = 4.50$

Now comes the vital step, take the natural log of both sides:

$\ln\left(e^{60k}\right) = \ln 4.50 \quad \text{or}$

$60k = \ln 4.50 \quad \text{or}$

$k = \dfrac{\ln 4.50}{60} = 1.504077/60 = .02507$

22. Completing the problem

$$P(t) = 100e^{.02507t}$$

To find the doubling time, we want to know the value of t for which $P(t) = 2P(0) = 200$, that is,

$100e^{.02507t} = 200 \quad \text{or}$

$e^{.02507t} = 2$

Taking the natural log of both sides:

$\ln\left(e^{.02507t}\right) = \ln 2 \quad \text{or}$

$.02507t = \ln 2 \quad \text{or}$

$$t = \frac{\ln 2}{.02507} = .693147/.02507 = 27.65$$

So the population doubles every 27.65 minutes.

23. Exponential Decay

- Exponential decay problems are very similar to exponential growth problems.
- They obey the formula

$$P(t) = P(0)e^{kt}$$

where the constant k is negative
- because as time increases, there is less of the material present
- A common example of exponential decay is radioactive materials
- the half-life is the time it takes for half of the material to decay into other elements

24. Exponential Decay Problem

The half-life of radioactive carbon–14 (^{14}C) is approximately 5730 years.

(Your book uses the figure 5750, but most sources cite 5730 as the actual value.)

That is, for every gram of ^{14}C today, if you wait 5730 years, you will have only 1/2 grams left.

If you have 50 grams of ^{14}C today, how many grams will be left after 100 years?

Let $P(t)$ be the amount of ^{14}C present at time t.

Exponential decay obeys the formula

$$P(t) = P(0)e^{kt}$$

where k is the decay constant.

25. Finding k

Decay Equation: $P(t) = P(0)e^{kt}$

We are told that the half-life is 5730 years:

$P(5730) = \frac{1}{2}P(0)$

Plugging these values into the decay equation gives

$P(5730) = P(0)e^{k5730} = \frac{1}{2}P(0)$ or

$e^{5730k} = \frac{1}{2}$

Take the natural log of both sides:

$\ln\left(e^{5730k}\right) = \ln 0.5$ or $5730k = \ln 0.5$ or

$k = \dfrac{\ln 0.5}{5730} = -0.693147/5730 = -.000120968$

Note that

1. k is negative, indicating decay

2. We did not need the actual value of $P(0)$ (50 gm in this case) to determine k

26. Completing the problem

$$P(t) = 50e^{-.000120968t}$$

To find the amount when $t = 100$, just plug $t = 100$ into the formula

$P(100) = 50e^{-.000120968 \cdot 100}$

$= 50e^{-.0120968}$

$= 50 \cdot .987976$

$= 49.4$ grams

Carbon dating, using ^{14}C, is widely used today to estimate the age of fossils

The idea is to calculate the amount of radioactive ^{14}C in the fossil todday, compare it with the amount that would be present in a living specimen, and thereby estimate the time required to achieve this percentage of decay.

27. Compound Interest

Suppose you invest \$1000 in a bank which pays 5% interest.

That means at the end of a year your account is worth

$$1000 + (.05)1000 = (1.05)1000 = 1050$$

The fifty dollars is the interest you earn for letting the bank keep your money for a year.

If the bank compunds your interest monthly,

- they break the year into 12 periods
- you make an interest rate of $\frac{.05}{12}$ over each of these periods
- each month you make interest on the interest accumulated in the previous months

28. Compound Interest Cont

The total principal after one year is

$1000(1 + .05/12)^{12} = 1000 \cdot 1.05116 = 1051.16$

You made an extra \$1.16 because interest was compounded

Well, it can amount to a great amount of money if the time period is long, like 30, 50, even a hundred years.

Interest can be compunded daily, hourly, every second, even continuously.

29. General Formula

The principal after a year if P_0 is invested at an annual interest rate r compounded over n time-intervals:

$$\boxed{P = P_0\left(1+\frac{r}{n}\right)^n}$$

For $r = .05$ we have the following table

periods	n	$\left(1+\frac{r}{n}\right)^n$
year	1	.05
month	12	1.051161898
day	365	1.051267496
hour	8760	1.051270946
minute	525600	1.051271094
second	31536000	1.0512710963

30. Continuous Compound Interest

It turns out that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

Based on this fact, the principal after time t if P_0 is invested continuously at an annual interest rate r is

$$\boxed{P = P_0 e^{rt}}$$

With $r = 0.5$, and $P_0 = 1000$, after one year

$$P(1) = 1000e^{.05} = 1000 \cdot 1.051271096$$

This is essentially the same value obtained when interest was compounded every second.