Substitution

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1. Five Basic Integration Formulas

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
- $\displaystyle\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int f(x) + g(x)\,dx = \int f(x) + \,dx + \int g(x)\,dx$
- $\displaystyle\int k \cdot f(x)\,dx = k \int f(x)\,dx$

  What do you do if you want to integrate a function which does not have the form of the first three types?

2. The Chain Rule Backwards

Even though there is no Chain Rule for integrals, you can sometimes work backwards.

Integrate: $\displaystyle\int \sqrt{2x+1}\,dx$

We know ; $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$

So it seems reasonable to try, as a first solution to $F(x) = \displaystyle\int \sqrt{2x+1}\,dx$

$F_g(x) = \dfrac{2}{3}(2x+1)^{3/2}$

("g" stands for "guess")

(and ignoring the $+C$ until the very end)

3. Fixing our guess

For the anti-derivative of $\sqrt{2x+1}$, we made the guess

$$F_g(x) = \frac{2}{3}(2x+1)^{3/2}$$

To test whether $F_g(x)$ is really the anti-derivative of $f(x) = (2x+1)^{1/2}$, differentiate. By the Chain Rule

$$\begin{aligned} F_g'(x) &= \frac{2}{3}\cdot\frac{3}{2}(2x+1)^{1/2}\cdot 2 \\ &= 2\,(2x+1)^{1/2} = 2\,f(x) \end{aligned}$$

We are off by a factor of 2.

We can correct for this factor of 2 by multiplying $F_g(x)$ by $\frac{1}{2}$. The correct anti-derivative is

$$F(x) = \frac{1}{2}\frac{2}{3}(2x+1)^{3/2} = \frac{1}{3}(2x+1)^{3/2}\ +C$$

You can check this answer by differentiation.

4. The Method of Substitution

There is a method, called substitution, which eliminates the guess work.

(1) Find an expression $g(x)$
   (a) inside a parenthesis
   (b) inside a square root
   (c) in a denominator
   (d) in an exponent
   for the substitution.
(2) Write $u = g(x)$ and replace every occurence of this expression in the integral by u.

5. Substitution Continued

(3) Separate $\dfrac{du}{dx} = g'(x)$ into

$$du = g'(x)\,dx$$

$$\text{or } dx = \frac{du}{g'(x)}$$

(4) Substitute $\frac{du}{g'(x)}$ for dx in the integral

(5) Hope for a miracle

(6) Once the variable x disappears, integrate the new function with respect to variable u

(7) Reverse the substitution so that your answer is in terms of x

6. Eliminating the guesswork

We return to our previous example

$$\int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx$$

Substitute $u = g(x) = 2x + 1$

$$\frac{du}{dx} = 2$$

$$du = 2\,dx \quad \text{or} \quad dx = \frac{du}{2}$$

$$\text{So} = \int (2x+1)^{1/2}\,dx = \int u^{1/2}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

7. An example with a miracle

Integrate $\displaystyle\int \frac{x+1}{(x^2+2x+3)^4}\,dx$

Substitute $u = x^2 + 2x + 3$

$\frac{du}{dx} = 2x + 2$

$$du = (2x+2)\,dx \quad \text{or} \quad dx = \frac{du}{2x+2}$$

$$\text{So } \int \frac{x+1}{(x^2+2x+3)^4}\,dx = \int \frac{x+1}{u^4}\,\frac{du}{2x+2}$$

$$= \int \frac{x+1}{u^4}\,\frac{du}{2(x+1)} = \frac{1}{2}\int u^{-4}\,du \quad \textit{the miracle}$$

$$= \frac{1}{2}\,\frac{u^{-3}}{-3} + C = -\frac{1}{6}u^{-3} + C \quad \text{reversing subst}$$

$$= -\frac{1}{6}(x^2+2x+3)^{-3} + C$$

8. An example with e^x

$$\text{Integrate } \int x^2\,e^{x^3+4}\,dx$$

Substitute $u = x^3 + 4$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx \quad \text{or} \quad dx = \frac{du}{3x^2}$$

$$\text{So } \int x^2\,e^{x^3+4}\,dx = \int x^2 e^u\,\frac{du}{3x^2} \quad \textit{the miracle}$$

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3+4} + C$$

9. Trickier Example

$$\text{Integrate } \int \frac{e^x}{e^x+1}\,dx$$

Substitute $u = e^x + 1$

$$\frac{du}{dx} = e^x$$

$du = e^x dx \quad \text{or} \quad dx = \frac{du}{e^x}$

So $\int \frac{e^x}{e^x + 1}\, dx = \int \frac{e^x}{u} \frac{du}{e^x}$ *the miracle*

$= \int \frac{1}{u}\, du = \ln u + C = \ln(e^x + 1) + C$

10. Final Example

Integrate $\int \frac{(5 + \ln x)^2}{x}\, dx$

Substitute $u = 5 + \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\, dx \quad \text{or} \quad dx = x\, du$

So $\int \frac{(5 + \ln x)^2}{x}\, dx = \int \frac{u^2}{x}\, x\, du$ *the miracle*

$= \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}(5 + \ln x)^3 + C$