Behavior of Exponential, Power, and Logarithmic Functions

This brief note gives information about the relative growth of exponential functions, power functions, and logarithmic functions. The basic result is: in a race to infinity, exponential functions come in first, followed by power functions, and logarithmic functions are in last place. Polynomials are just combinations of power functions and therefore find themselves in second place, behind exponential functions and ahead of logarithmic functions.

How do we state these ideas mathematically? The general form of an exponential function is e^{ax} where a is a positive constant. A power function looks like x^b, where b is a positive constant. As it turns out, no matter how large we make the constant b and no matter how close to zero we make the constant a, eventually the function e^{ax} will over take x^b. For example, $e^{0.00001x}$ will eventually overtake the function $x^{46000000000}$. (Can you figure out when?) We describe the behavior of the exponential function versus the power function as a limit:

$$\lim_{x\to\infty} \frac{x^b}{e^{ax}} = 0. \tag{1}$$

The behavior of the power function x^a versus the logarithmic function $(\ln x)^b$ is expressed by the limit

$$\lim_{x\to\infty} \frac{(\ln x)^b}{x^a} = 0. \tag{2}$$

A simple proof of statment (2) may be given directly from the integral definition of the natural logarithm. Let c be any positive constant. (The value of c will be determined later.) For all $t \geq 1$ we know that $t^c \geq 1$ and so

$$\frac{1}{t} = t^{-1} \leq t^{-1}t^c = t^{c-1}.$$

Considering the areas under these two curves, it follows that for $x > 1$

$$\int_1^x \frac{1}{t}\,dt \leq \int_1^x t^{c-1}dt.$$

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Performing the integration we find

$$0 < \ln x \le \frac{x^c - 1}{c} < \frac{x^c}{c}.$$

Raising both sides to the power b we get

$$0 < (\ln x)^b < \frac{x^{bc}}{c^b}.$$

Now divide by x^a to get our final inequality:

$$0 < \frac{(\ln x)^b}{x^a} < \frac{x^{bc-a}}{c^b}.$$

This inequality holds for any positive constant c. It is now time to select an appropriate value for c. Choose $c = \dfrac{a}{2b}$. Then x^{bc-a} becomes $x^{-a/2} = \frac{1}{x^{a/2}}$, which approaches 0 as x approaches infinity. By the Squeeze Theorem, it follows that $\dfrac{(\ln x)^b}{x^a}$ must also tend to 0 as $x \to \infty$. This proves statement (2).

Statement (1) follows easily from (2). Just make the change of variable $x = \ln t$. Then

$$\frac{x^b}{e^{ax}} = \frac{x^b}{(e^x)^a} = \frac{(\ln t)^b}{\left(e^{\ln t}\right)^a} = \frac{(\ln t)^b}{t^a},$$

which approaches 0 as x (and hence $t = e^x$) goes to infinity. This proves statement (1).