MODULE 1

Infinity

"Eternity?" said Frankie Lee, with a voice as cold as ice.

"That's right," said Judas, "eternity, though some call it paradise." – Bob Dylan

The essence of mathematics lies in its freedom. – Georg Cantor

1. The 999 Question

1.1. Decimal Expansions and the 999 Question.

We begin with

Question 999: | Does $.999\cdots = 1?$ |

Here are some possible answers:

- "Yes, they are equal."
- "No, $.999\cdots$ is not a number!"
- "No, these two numbers are very close to each other, but not identical."
- "I don't understand the question."
- "I know what the question is asking, but do not know the answer."

It is my experience that most students are reluctant to answer Question 999 affirmatively.

Professor Flappenjaw is discussing this problem with two students, Ernest Noddsoff and Hedda Earful.

Ernest: First of all, what exactly is $.999\cdots$? Is it a number?

Professor: It is a repeating decimal, like $\frac{1}{3} = 0.333\cdots$. The decimal number $.999\cdots$ lies at some point on the number line. Question 999 simply asks: Is this also the point corresponding to the number 1?

Ernest: Well, they don't look the same.

Professor: That's right. But then, the repeating decimal $.333\cdots$ and the fraction $\frac{1}{3}$ don't look the same either. The rational number $\frac{1}{2}$ has infinitely many representations

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots .$$

Yet the fact that all these different looking fractions are the same number doesn't bother anyone. For that matter, $1 + 1$ and 2 don't look the same, but they are.

Hedda: It seems to me that on the number line $.999\cdots$ will never quite make it to the point 1. So it must lie just a little to the left of 1.

Professor: This is really one of the fundamental questions of mathematical analysis: What do you mean by "just a little?" Do you include zero? For if the distance between $.999\cdots$ and 1 is zero, doesn't this really mean that the two numbers are the same? Or does "just a little" mean a small positive number, but not zero?

Hedda: By "a little bit" I certainly don't mean to include zero. I would never say to dinner guests "There is a little bit of pie left. Does anyone want it?" if all the pie were gone.

Ernest: You might have some angry guests if you did.

Professor: So Question 999 boils down to the question: If we mark $.999\cdots$ and 1 on the number line, will there be a little tiny space between them or will they be exactly the same point.

Ernest: You know what this reminds me of? I watched my old Dodge van turn one hundred thousand miles last summer. It was real cool to watch all those 9's turn into all those 0's. But everyone knows that having $99,999$ on your odometer is not the same as when it reads $100,000$. In the 999 Question the difference between $.999\cdots$ and $1.000\cdots$ reminds me of the difference between seeing all 9's on an odometer and seeing all 0's.

Hedda: Except that you would have to have a pretty large dashboard to display all those digits.

Professor: You are right, Ernest. The difference between 99999 and 100000 is exactly 1 mile. Have you ever seen odometers—some trip odometers are like this—which display tenths of a mile? Then the difference between 999.9 and 1000 is one tenth of a mile. If your odometer displayed hundredths of a mile, then the difference between 999.99 and 1000 would be one hundredth of a mile. And so on; every time you add another digit, the difference decreases by a factor of 10.

Ernest: I got an idea. Maybe on the number line, the point .999 · · · is the point *just before* the point corresponding to the number 1.

Professor: Do you think it is possible for two points on a line to touch each other, like beads on a necklace?

Ernest: Sure, why not?

Hedda: If you have two different points on a line, no matter how close they are, isn't their midpoint always between them? Which means they can't actually be touching each other.

Ernest: I hadn't thought of that.

Professor: Hedda's right, Ernest. And if .999 · · · and 1 really are two different points on the number line, then how would you write the decimal expansion of the midpoint between them?

Hedda: I see what you're driving at. Suppose you examine the digits of the midpoint one at a time. Clearly the midpoint begins with a bunch of 9's after the decimal point. If you ever use a digit other than 9, then the midpoint will be less than .999 · · ·, and that ain't right. On the other hand, if you keep using 9's forever, then the midpoint will be identical with the endpoint .999 · · · and that ain't right either.

Ernest: Why can't digit number infinity of the midpoint be a 5?

Professor: What you're suggesting is that at the end of infinitely many 9's we write the digit 5. The problem is that you never come to the end of all the 9's in .999 · · ·. There is no *last* digit.

Ernest: All this is making my head hurt. Can't you just tell us the answer, Professor?

Professor: Hedda was on the right track, but let me give you a slightly more "round-about" answer.

Ernest: It seems like all your explanations are round-about answers, Professor.

Professor: Spare me the comments, Ernest. As you know from grade school, the real numbers have a natural order, called 'less than' ($<$). Without going into too many details, one of the basic properties of 'less than' is that if we have any two real numbers a and b, then precisely one of the following relations hold: (i) $a < b$, (ii) $a = b$, or (iii) $b < a$. This is called the law of **trichotomy**.

Ernest: I can see what's coming. You want to make a equal to $.999\cdots$ and b equal to the number 1.

Professor: Exactly! When we compare $a = .999\cdots$ to $b = 1$, then by the law of trichotomy, there is precisely one possiblity: (i) $.999\cdots < 1$, (ii) $.999\cdots = 1$, or (iii) $.999\cdots > 1$. We can rule out choice (iii) since no decimal that begins with 0.9 can exceed 1.0. Now what about choice (i)? Let's say, for the moment, that we believe choice (i) $.999\cdots < 1$. Call the difference between these two numbers $d = 1 - .999\cdots$ and write d as a decimal:

$$d = 0.d_1 d_2 d_3 \cdots .$$

Since we are assuming $d > 0$, not all the digits $d_1, d_2, d_3, \cdots$ are 0. Let's say d_n is the first nonzero digit. Then since d_n must be one of the digits 1 through 9, it follows that

$$d \geq 0.\underbrace{0\cdots 0}_{n-1} d_n \geq 0.0\cdots 01 = \frac{1}{10^n}.$$

On the other hand, the number $.999\cdots$ is certainly larger than the number obtained by truncating the 9's after you write n digits. That is,

$$.999\cdots > .\underbrace{999\cdots 9}_{n}.$$

From this we get

$$d = 1 - .999\cdots < 1 - .\underbrace{999\cdots 9}_{n} = \frac{1}{10^n}.$$

But how can the two inequalities $d \geq \frac{1}{10^n}$ and $d < \frac{1}{10^n}$ both be true? For example, if $n = 2$, we cannot have $d \geq \frac{1}{100}$ and $d < \frac{1}{100}$ at the same time.

Ernest: So we're screwed!

Professor: That's one way of putting it. But of course, the problem only arises if we believe choice (i), that is $.999\cdots < 1$, is true. So choice (i) must be false, as was choice (iii).

Hedda: Where does this leave us?

Professor: To paraphrase Sherlock Holmes, if we have three statements, one of which must be true, and we eliminate two of the statements as being false, then the remaining statement, no matter how improbable, must be true.

Hedda: Let me finish your reasoning, Professor. Since choices (i) and (iii) of the trichotomy principle are false, the remaining choice (ii) has to be true. In other words, the answer to Question 999 is that the equation $.999 \cdots = 1$ is true.

Ernest: I like your Sherlock Holmes logic here, Professor. But tell me, wasn't Professor Moriarity, the arch nemesis of Sherlock Holmes, a mathematician?

Professor: Not all mathematicians are evil, Ernest.

Ernest: I was just joking, Professor. Seriously, though, there's something that still bothers me. If the real numbers are just infinite decimal expansions, then the equation $.999 \cdots = 1.000 \cdots$ means that the number 1 has two different decimal expansions. Do all numbers have two different decimal representations?

Professor: We can define the real numbers as infinite decimal expansions. If the expansion has, after an initial start, a repeating pattern such as $x = 5.6666666 \cdots$, then the number is a rational number, in this case $x = 5\frac{2}{3}$ or $\frac{17}{3}$. If the decimal expansion of x does not repeat, then x is an irrational number, like $\pi = 3.14159265 \cdots$. There is only one way to write the decimal expansion of an irrational number.

Ernest: So rational numbers have two different expansions?

Professor: No, that's not true either. The only way to write the fraction one third in decimal is $.333 \cdots$ using all 3's. On the other hand one fourth can be written as $.25$ and $.24999 \cdots$.

Hedda: Ye Gads, we're back to the 999 Question again.

1.1.1 Exercises

1. Here are two other arguments to support the claim that $.999 \cdots = 1$. Are they convincing?

Argument 1. Multiplying the equation $.333 \cdots = \frac{1}{3}$ by 3 gives $.999 \cdots = 3 \times \frac{1}{3} = 1$.

Argument 2. Let $x = 0.999\cdots$. To find $10x$ we just move the decimal point one place to the right. Subtracting the two equations

$$10x = 9.999\cdots$$
$$x = 0.999\cdots$$

gives $9x = 9$. Divide by 9 to get $x = 1$.

2. Using the technique in Argument 2, write the following repeating decimal numbers as fractions:

$$0.181818\cdots$$
$$2.272727\cdots$$
$$0.46534653\cdots$$
$$0.142857142857\cdots$$

3. Characterize all real numbers which have two different decimal expansions, like $.24999\cdots$ and $.25$.

4. [Challenge Problem] Find distinct decimal digits for the letters E, V, D, I, T, A, L, and K which gives the repeating decimal pattern:

$$\frac{EVE}{DID} = .TALKTALKTALK\cdots$$

Note that (1) the eight letters must all be different and (2) you must use the same digit for every occurrence of each letter, that is, for the two E's in "EVE" and the two D's in "DID."

$$\text{A simple cryptarithmetic problem is} \quad \begin{array}{r} AB \\ + CD \\ \hline EFG \end{array}$$

1.2. Zeno, Binary Expansions, and the 111 Question. A version of Question 999 first got air time over two thousand years ago thanks to a mathematical prankster named Zeno, who went around confusing people with his riddles. Zeno's Paradox, roughly translated, is

"In order to travel from your couch to the door, you must first get to the halfway point, from there you must travel to the second halfway point, then to the third halfway point, and so on. Since there are infinitely many halfway points, you can never reach the door."

Zeno, being male, expounded the ultimate "guy" belief: if you can't get off the couch, then you might as well lay back and watch some football. (Of course, somebody needs to bring the snacks.)

Zeno was correct in asserting that the distance from the couch to the door is an infinite sum:

$$L = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

But behind Zeno's Paradox is the assumption that the sum of infinitely many positive real numbers must add to infinity. Right? No, it's not right. Decimal expansions are infinite sums:

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \cdots$$

and no one these days thinks that the infinite sum on the right is infinity.

Pehaps the best way to understand Zeno's Paradox is to use binary (or base 2) notation. If each of the binary digits (or *bits*) $b_1, b_2, b_3, \ldots$ is either 0 or 1, then the binary number $0.b_1b_2b_2b_3 \cdots$ is defined to be

$$0.b_1b_2b_2b_3 \cdots \text{ (base 2)} \overset{\text{def}}{=} b_1 \cdot \frac{1}{2} + b_2 \cdot \frac{1}{4} + b_3 \cdot \frac{1}{8} + \cdots + b_n \cdot \frac{1}{2^n} + \cdots .$$

The positions after the binary point are powers of $\frac{1}{2}$, just as in decimal notation, positions after the decimal point are powers of $\frac{1}{10}$. Thus $L = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has the binary representation

$$L = 0.1111 \cdots \text{ (base 2)}.$$

Zeno's Paradox boils down to

Question 111: $\boxed{\text{Does } .111 \cdots = 1 \text{ (base 2)?}}$

A little thought shows us that Question 111 is the binary equivalent of Question 999 for decimals.

It is clear that the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ will never exceed one: when you get to the first halfway point, you have one half of the distance left to go; when you get to the second halfway point, you have one fourth of the original distance to go, etc. As an equation, this observation becomes

(Zeno's Approximation) $\qquad\qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$

Zeno's Approximation represents an infinite family of arithmetic facts:

$$\frac{1}{2} = 1 - \frac{1}{2}$$
$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$$
etc.

You cannot check them all. Happily, you don't have to. You only need to see how to get from each line in this family of equations to the next line. For example, to get the fourth line, add $\frac{1}{16}$ to the third line:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1 - \frac{1}{8} + \frac{1}{16}$$
$$= 1 - \frac{2}{16} + \frac{1}{16}$$
$$= 1 - \frac{1}{16}.$$

This process is called mathematical induction. It simply says that if you can always take one more step—and if you can take the first step—then you can walk forever.

The conclusion to be drawn from Zeno's Approximation is that we can make the sum $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$ really close to 1 by making n large enough. What happens when we add up all the terms out to infinity? The argument is exactly the same as the argument in the 999 problem. The sum never exceeds one. On the other hand, we get in trouble if we say the sum is a number r less than 1, since eventually—using enough terms—the sum will exceed r. The only conclusion left is that the sum is exactly one, Zeno was wrong, and people can get off the couch, at least if they want to.

1.1.2 Exercises

1. There are several ways to settle the 111 Question, just as there were several ways to settle Question 999. What are some algebraic ways to settle Zeno's paradox by actually computing the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$?

2. Explain how to compute cube roots with a calculator that has only a square root button (and no x^y key). [Hint: consider the binary expansion of $\frac{1}{3}$.]