

## 5. Geometric Series and Three Applications

**5.1. Geometric Series.** One unsettling thing about working with infinite sums is that it sometimes happens that you know that the sum is finite, but you don't know what its actual value is. This happens all the time in nature. We know that the number of leaves on all the trees in the world is a finite number, but no one knows what the exact count is. In mathematics, on the other hand, we like to be exact whenever possible. For example, it can be shown using calculus that the infinite sum

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

On the other hand, we know that the sum

$$T = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \cdots$$

is finite, since  $\frac{1}{n^3}$  is smaller than  $\frac{1}{n^2}$  and the sum of the reciprocals of the squares is finite. But we don't know the precise value of the sum  $T$ . In mathematics, when we don't know the exact value of a quantity, we give it a name. For example, we know the value of the positive solution to  $x^2 = 25$  is  $x = 5$ , but we do not know the exact value of the solution to  $x^2 = 7$ , so we give it a name, "the square root of 7," and a symbol,  $\sqrt{7}$ . If you are curious, the sum  $T$  is given the name  $\zeta(3)$ , where  $\zeta$  is the Greek letter zeta. [This information may come in handy if you are ever in a game show and the category is math trivia.]

There is a category of infinite sums which can be evaluated precisely.

**Definition 1.1** (geometric series). A **geometric series** is a series of the form

$$S = a + ar + ar^2 + ar^3 + ar^4 + \cdots .$$

Here the first term is  $a$  and we obtain each of the other terms by multiplying the term preceding it by the ratio  $r$ . To find the sum  $S$ , multiply the geometric series by  $r$ :

$$rS = ar + ar^2 + ar^3 + ar^4 + \cdots .$$

Now subtract this last equation from the equation for a geometric series, to get

$$S - rS = a.$$

By a miracle, all the terms on the right hand side cancel except for  $a$ . Since we can factor  $S - rS = (1 - r)S$ , we get an exact value for  $S$ :

$$S = \frac{a}{1 - r}.$$

**Problem 1.** Use this formula to evaluate

$$S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$$

The first term is  $a = \frac{3}{10}$  and the common ratio is  $\frac{1}{10}$ , so the sum is

$$S = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{3}{9} = \frac{1}{3}.$$

In decimal notation, this says

$$.333333 \cdots = \frac{1}{3},$$

a well known arithmetic fact.

**Problem 2.** Use the geometric series formula to evaluate

$$S = 1 + 2 + 4 + 8 + 16 + \cdots$$

The first term is  $a = 1$  and the common ratio is 2, so the sum is  $S = \frac{1}{1 - 2} = -1$ , a completely bogus answer. What happened? For the first question we expected an answer and we got one—happily the right answer. For the second question we should never expect to get an answer. The sum  $1 + 2 + 4 + \cdots$  will continue to grow to infinity. The formula for the sum of a geometric series cannot be used with reckless abandon, or else it will produce goofy results, like the answer  $-1$  to Question 2. In order to get correct results, we can only use the formula for a geometric series when the ratio  $r$  lies in the range  $-1 < r < 1$ .

**Theorem 1** (Geometric Series Formula). If  $-1 < r < 1$ , then

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1 - r}.$$

Once mathematicians develop a formula, they want to use it. The following three applications tie up three threads woven throughout this first chapter.

**5.2. Application 1: The 999 Question.** We return to the question which started this chapter

$$\text{Does } .999 \cdots = 1?$$

The decimal notation disguises the fact that the 999 Question is really asking a question about infinite sums:

$$\text{Does } \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1?$$

But the sum  $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$  is a geometric series with first term  $a = \frac{9}{10}$  and ratio  $r = \frac{1}{10}$ . The ratio  $r$  is between  $-1$  and  $1$ , so we can use the formula for a geometric series:

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

Done. Finis. The 999 Question is settled once and for all.

**5.3. Application 2: How much space do the rational numbers occupy?** We saw in the section on Infinite Hotels that the positive rational numbers can be counted  $r_1, r_2, r_3, r_4, \dots$ . Recall that the way to count the rationals uses the diagonal method for solving the Infinite Bus problem, except we have to be careful not to count fractions such as  $\frac{1}{2}$  more than once—the trick is to ignore equivalent representations  $\frac{2}{4}, \frac{3}{6}$ , etc. Using the diagonal method, our list of rationals begins:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{1}{5}, \frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \dots$$

Take the first rational  $r_1 = \frac{1}{1}$  and cover it with a blanket of length  $\frac{1}{10}$ . Put  $r_1$  in the middle of this blanket, so that the blanket lies on the interval  $(.95, 1.05)$ . Put a blanket of length  $\frac{1}{100}$  around the second rational in the list  $r_2 = \frac{2}{1}$ . If we center  $r_2$ , this blanket lies on the interval  $(1.995, 2.005)$ . Now put a blanket of length  $\frac{1}{1000}$  around the third rational number  $r_3 = \frac{1}{2}$ . Continue this process. At the  $n$ th stage, place a blanket of length  $\frac{1}{10^n}$  about the  $n$ th rational  $r_n$ . It is clear that the size of the rational numbers cannot exceed the total length of the blankets used to cover them all. (Never mind that some rationals will lie inside more than one blanket; this just keeps them extra warm. The point is that all rationals are covered by at least one blanket.) The total length of the blankets (Never mind also that the blankets may overlap) is

$$L = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots + \frac{1}{10^n} + \cdots.$$

Note that this sum is a geometric series, with first term  $a = \frac{1}{10}$  and common ratio  $r = \frac{1}{10}$ . By the formula,

$$L = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{10} \cdot \frac{10}{9} = \frac{1}{9}.$$

So the length of all the rationals is less than  $\frac{1}{9}$ .

We now re-examine what we did and realize that the first blanket was way bigger than necessary to cover a single point. So we use the same strategy of covering our collection of rationals with blankets, one at a time, each time using a blanket one tenth as large as the

previous blanket, except this time we start with a blanket of size  $\frac{1}{100}$ . The positive rationals are now swathed by blankets whose total length is less than

$$L' = \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots .$$

By the formula for the sum of geometric series,

$$L' = \frac{\frac{1}{100}}{1 - \frac{1}{10}} = \frac{\frac{1}{100}}{\frac{9}{10}} = \frac{1}{100} \frac{10}{9} = \frac{1}{90}.$$

So the length of the positive rationals is less than  $\frac{1}{90}$ .

We repeat the process again, this time beginning with a blanket of size  $\frac{1}{1000}$ , so that the total length of all the blankets becomes, by the formula for geometric series,  $\frac{1}{900}$ . Since we can keep this up, we are forced to conclude that the total length of the rational numbers can be made smaller than any positive number, no matter how tiny. In other words, the total length of the rational numbers must be zero.

**5.4. Application 3: The Cantor Set.** What is the length of the Cantor Set? The easiest way to answer this question is to compute the length of the “middle third” intervals which are removed at each stage of creating the Cantor Set. The first middle third has length  $\frac{1}{3}$ . The second middle third has length  $\frac{1}{9}$ , but since we remove the middle thirds of two subintervals, the total length removed at the second stage is  $\frac{2}{9}$ . At the third stage we remove four middle thirds, each of length  $\frac{1}{27}$ , so the total length removed is  $\frac{4}{27}$ . Since these middle thirds do not overlap, we see that the total length of the intervals removed is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots .$$

Once again, this is a geometric series, with first term  $a = \frac{1}{3}$  and common ratio  $r = \frac{2}{3}$ . By the formula, the total length  $L$  of all these middle thirds is

$$L = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1.$$

If we begin with the unit interval  $[0, 1]$  and remove intervals whose total length is 1, the points that remain—in this case, the Cantor set—must have length  $1 - 1$ , or 0. The answer to Question 1 at the end of the Cantor Set Section demonstrates that there are as many points in the Cantor set as there points in the interval  $[0, 1]$ . It boggles the mind to think that they all fit in a space whose total length is zero.

### 1.5 Exercises

1. Evaluate: (a)  $3 + \frac{6}{7} + \frac{12}{49} + \frac{24}{343} + \cdots$  (b)  $3 - \frac{6}{7} + \frac{12}{49} - \frac{24}{343} + \cdots$

2. Repeating decimals turn out to be the rational numbers. For example,  $\frac{1}{3} = 0.333\cdots$ . We sometimes write a line over the repeating part, so  $\frac{1}{3} = .\overline{3}$ . What fractions (in lowest terms) have the following decimal expansion:

(a)  $1.\overline{2}$  (b)  $0.\overline{09}$  (c)  $0.\overline{627}$  (d)  $0.\overline{076923}$

What fractions (in lowest terms) have the following binary expansion:

(a)  $0.\overline{001}$  (b)  $0.\overline{00101}$  (c)  $0.\overline{1001}$  (d)  $0.\overline{10101}$

3. Compute the repeating part of the decimal expansion of  $1/7$ . You should get 142857. Multiply the number  $x = 142857$  by 2, 3, 4, 5, and 6. Compare the string of digits in each product with the original string of 6 digits.

4. Use geometric series to solve the 111 Question, related to Zeno's Paradox: Does  $0.111$  (base 2)  $= 1$ ?

5. If a ball is dropped from a 12 foot roof and bounces two thirds as high on each successive bounce, find the total number of feet that the ball will travel, including all the bounces.

6. Stephen Hawking is reported to have said that his editor told him that in writing popular books about math and physics you lose one half your audience for every equation placed in the book. (What a surprise: equations turn off the general population.) From an original potential population of 12 million readers, how many readers will read his book "A Brief History of Time" if there are 12 equations scattered throughout the book?

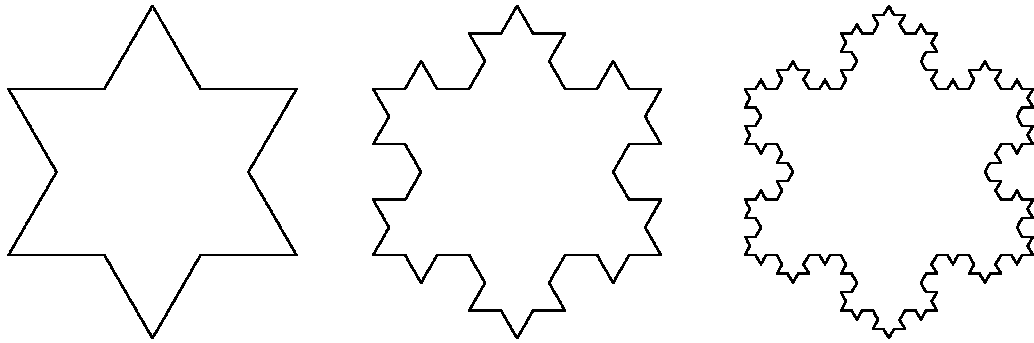
7. We return to Exercise 2 of the last section, about the two dimensional Cantor Set obtained by successively removing the middle fourths from the remaining equilateral triangles in a diagram, starting with one equilateral triangle whose side length is 1. Let  $A$  be the area of the original equilateral triangle. Compute the value of  $A$  by splitting the triangle into right triangles. At the first stage the area of the first middle fourth is  $\frac{1}{4}A$ . At stage 2, we have three triangles whose middle fourth each has area  $\frac{1}{16}A$ . At the next stage we get nine triangles whose middle fourth each has area  $\frac{1}{64}A$ . Evaluate the sum

$$\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots$$

What is the total area of all the middle fourths that have been removed? What is the total area of the set that's left after repeatedly removing the middle fourths from the remaining triangles in the diagram?

8. Start with an equilateral triangle of side length 1. On the middle third of each side, attach an equilateral triangle of length  $1/3$ , pointing outwards, and then erase the original middle third of that side. You should now have a 12-sided polygon with 6 vertices. Each side

has length  $1/3$ . On the middle third of each of the 12 sides, attach an equilateral triangle of length  $1/9$  and then remove the original middle third. The resulting figure has 36 sides, each of length  $1/9$  and 18 vertices. Continue the process forever. The resulting figure is called a **snowflake curve**. The first three iterations of the snowflake curve are drawn below:



Show that this curve has the following remarkable properties: (i) its area is finite and (ii) its perimeter is infinite. You can paint the inside of the snowflake curve, but it would take *forever* to walk around its boundary.