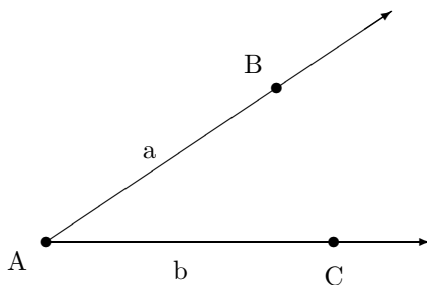


6. Angles

Two rays a and b are called **coterminal** if they have the same endpoint. If this common endpoint is A , then there must be points B and C such that

$$a = \overrightarrow{AB} \quad \text{and} \quad b = \overrightarrow{AC}.$$

The union of the two coterminal rays \overrightarrow{AB} and \overrightarrow{AC} is called the **angle** $\angle BAC$.



Each angle has a **measure** $m\angle BAC$. We also write $m\angle BAC$ as $m(a, b)$ to indicate that angle measure is a kind of “distance” between two coterminal rays a and b . Angle measurement must satisfy the following four properties:

Angle Measurement Axioms

Axiom 1. $0 \leq m(a, b) \leq 180$.

Axiom 2. $m(a, b) = 0$ if and only if $a = b$. [zero property]

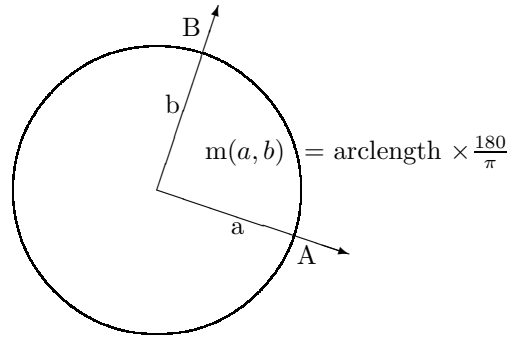
Axiom 3. $m(a, b) = 180$ if and only if a and b are opposite rays.

Axiom 4. $m(a, b) = m(b, a)$ [symmetry].

Example 4.1. Consider the model of the x - y plane. Given any pair of rays a and b whose common endpoint is the origin $(0,0)$. Each pair of rays a and b will intersect the unit circle

$$x^2 + y^2 = 1$$

at unique points A and B . We define the angle measure $m(a, b)$ to be the length of the shortest arc connecting A and B times $\frac{180}{\pi}$.



Given three coterminal rays a , b , and c , we use angle measurement to define precisely what we mean when we say that ray b is between rays a and c , written $a - b - c$.

Definition 4.1. $a - b - c$ if and only if a , b , c are three distinct coterminal rays and

$$m(a, b) + m(b, c) = m(a, c).$$

Betweenness Axiom for Rays. If three distinct coterminal rays a , b , and c , all lie in the same closed halfplane whose edge contains one of them, then a betweenness relation $a - b - c$, $a - c - b$, or $b - a - c$ holds.

Theorem 4.1. [unique middle for rays] If $a - b - c$ is true, then $a - c - b$ and $b - a - c$ are both false.

Notice the similarity between the definition of betweenness of points and betweenness of rays. We can easily modify our proof of the Unique Middle Theorem (Theorem ??) for collinear points to prove the Unique Middle Theorem for coterminal rays. All we do is “search and replace” the words

- point \rightarrow ray
- distance $d \rightarrow$ angle measurement m
- collinear \rightarrow coterminal
- upper case $A, B, C \rightarrow$ lower case a, b, c

Finally, we need an axiom which says we can make angles of any measure between 0 and 180. One technicality is that we want to be able to construct our angle on either side of the line containing one of the sides of the angle.

Protractor Axiom. Given a ray a , a halfplane H whose edge contains a , and a real number s in the interval $0 \leq s \leq 180$. Then there is a ray r with the same endpoint as a lying in the halfplane H such that

$$m(a, r) = s.$$

Some terminology: Angle $\angle ABC$ is a(n)

right angle if $m\angle ABC = 90$

acute angle if $0 < m\angle ABC < 90$

obtuse angle if $90 < m\angle ABC < 180$

zero angle if $m\angle ABC = 0$

straight angle if $m\angle ABC = 180$

proper angle if $0 < m\angle ABC < 180$

One way to remember how to distinguish an acute angle from an obtuse angle is to recall that in English, “acute” means sharp (an acute pain is a sharp pain) while “obtuse” means dull (an obtuse fellow is a dull person). An arrowhead whose tip forms an acute angle is sharp and dangerous, while an arrowhead whose tip forms an obtuse angle is not likely to penetrate skin.

The Protractor Axiom can be used to prove the following theorem about angle bisectors:

Theorem 4.2. [angle bisector] Given a nonzero angle $\angle BAC$. There is a unique ray \overrightarrow{AD} between \overrightarrow{AB} and \overrightarrow{AC} , called the **angle bisector** of $\angle BAC$, such that

$$m\angle BAD = m\angle DAC.$$

4.6 Exercises

1. Given three coterminal rays a , b , and c , is it always the case that at least one ray is between the other two? Hint: choose the rays so that $m(a, b) = m(a, c) = m(b, c) = 120$.
2. Is the ray x in the protractor axiom unique?
3. Prove Theorem 4.1.
4. Prove Theorem 4.2.

5. How can we define the interior of an angle $\angle BAC$? Is it true that an angle separates the plane into three distinct parts: (i) the angle itself, (ii) the interior of the angle, and (iii) the exterior of the angle?

7. Triangles

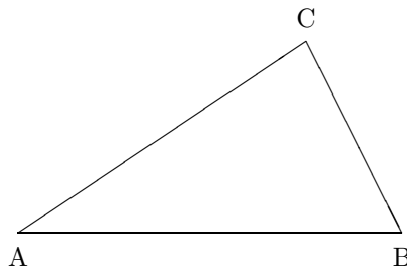
Professor Flappenjaw: Today we're studying triangles. Who can tell me what a triangle is?

Ernest: You need three points A , B , and C .

Professor: Is a triangle just these three points?

Chris: No, you gotta connect them to each other.

She draws on the board:



Professor: I see. So the triangle $\triangle ABC$ is just the union of the three segments

$$\overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

Ernest: Sweet.

Professor: And you could use any three points to make a triangle?

Joshua: As long as they're distinct, why not?

Professor: What if the three points all lie on the same line?

Ernest: You'd get a really flat triangle.

Professor: Do you want to even call such a thing a triangle?

Joshua: [grudgingly] I guess not.

Professor: Triangle $\triangle ABC$ is $\overline{AB} \cup \overline{BC} \cup \overline{AC}$, provided that A , B , and C are distinct and non-collinear. Each of the three noncollinear points is called a **vertex** of the triangle. Each segment is called a **side** of the triangle and each of the angles $\angle BAC$, $\angle ABC$, and $\angle ACB$ is called an **angle of the triangle**. These are often written more simply as $\angle A$, $\angle B$, and $\angle C$.

Ernest: It's easy to identify the angles. Angle A is the angle at vertex A . But it's harder to identify the sides.

Professor: Maybe this will help. Take triangle ABC . We call side \overline{BC} the side **opposite** $\angle A$. Notice that the letters B and C are the two vertices of the triangle other than A .

Chris: So to find the side opposite angle C , we delete C from the vertices A, B, C to get segment \overline{AB} .

Professor: Exactly. Conversely, angle $\angle A$ is the angle **opposite** side \overline{BC} .

Chris: In this case A is the letter missing from the endpoints of the segment, B and C .

Professor: Right. Tell me, can the side lengths be any three numbers we choose?

Ernest: As long as they're positive, why not?

Professor: Can you draw a triangle with sides of length 2, 3, and 7?

Matthew: No way, the 7 is too large.

Sarah: I see what the problem is. The triangle inequality says that the sum of two sides—in this case $2 + 3 = 5$ —must always be larger than the third side.

Ernest: Duh, that's why it's called the *triangle* inequality.

Chris: It's kind of strange that you can use any three pairs of points for the vertices—well, as long as they are not collinear—but you can't just use any three values for the lengths of the sides.

Professor: Interesting observation. The restrictions imposed by the triangle inequality, that is, that the sum of any two sides must always exceed the third, seems way more restrictive. It makes you think that side length is not the optimal way to describe a triangle. Can you think of any other way to generate triangles?

Sarah: If you draw three random lines, then the region inside them is a triangle.

Professor: Always?

Sarah: I guess it doesn't work if two of the lines are parallel. But then again, who would expect that random lines would turn out to be parallel?

Matthew: They couldn't all meet in one common point, either, or else the triangle would be a single point.

Professor: Exactly. Now, if I have three positive real numbers whose values do not violate the triangle inequality, that is, the sum of any two of them is greater than the third, then can I always draw a triangle whose lengths are these three numbers?

Ernest: I can't see any reason why you can't, but I can't see any way to prove why you always can.

Professor: Let me ask a slightly different question. Suppose I wanted a triangle $\triangle ABC$ where the measure of angle $\angle B$ is, say 67, and the lengths AB and BC are 7 and 29. Can you always draw such a triangle?

Chris: Well, you could use the Protractor Axiom to construct the 67 degree angle.

Joshua: And then you could use the Ruler Axiom to make the sides the required lengths.

Ernest: Thank God for Axioms.

Professor: Here's my final question to think about for the next class. Suppose that Joshua and Sarah each make a triangle whose sides have length 7 and 29, where the in-between angle measures 67. Will their two triangles look exactly the same?

4.7 Exercises

1. Draw a triangle $\triangle ABC$. Now draw a triangle CDE , where E lies on \overline{AC} and $C - B - D$. Which angle of $\triangle CDE$ is congruent to an angle of $\triangle ABC$.
2. Try do this problem without drawing a picture. Fill-in the blanks. In triangle $\triangle PQR$,
 - (a) segment \overline{RQ} is opposite _____.
 - (b) angle $\angle PRQ$ is opposite _____.
 - (c) segment \overline{QP} is opposite _____.