

## 8. Triangles

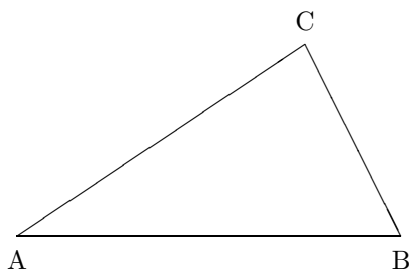
Professor Flappenjaw: Today we're studying triangles. Who can tell me what a triangle is?

Ernest: You need three points  $A$ ,  $B$ , and  $C$ .

Professor: Is a triangle just these three points?

Chris: No, you gotta connect them to each other.

*She draws on the board:*



Professor: I see. So the triangle  $\triangle ABC$  is just the union of the three segments

$$\overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

Ernest: Sweet.

Professor: And you could use any three points to make a triangle?

Joshua: As long as they're distinct, why not?

Professor: What if the three points all lie on the same line?

Ernest: You'd get a really flat triangle.

Professor: Do you want to even call such a thing a triangle?

Joshua: [grudgingly] I guess not.

Professor: Triangle  $\triangle ABC$  is  $\overline{AB} \cup \overline{BC} \cup \overline{AC}$ , provided that  $A$ ,  $B$ , and  $C$  are distinct and non-collinear. Each of the three noncollinear points is called a **vertex** of the triangle. Each segment is called a **side** of the triangle and each of the angles  $\angle BAC$ ,  $\angle ABC$ , and  $\angle ACB$  is called an **angle of the triangle**. These are often written more simply as  $\angle A$ ,  $\angle B$ , and  $\angle C$ .

Ernest: It's easy to identify the angles. Angle  $A$  is the angle at vertex  $A$ . But it's harder to identify the sides.

Professor: Maybe this will help. Take triangle  $ABC$ . We call side  $\overline{BC}$  the side **opposite**  $\angle A$ . Notice that the letters  $B$  and  $C$  are the two vertices of the triangle other than  $A$ .

Chris: So to find the side opposite angle  $C$ , we delete  $C$  from the vertices  $A, B, C$  to get segment  $\overline{AB}$ .

Professor: Exactly. Conversely, angle  $\angle A$  is the angle **opposite** side  $\overline{BC}$ .

Chris: In this case  $A$  is the letter missing from the endpoints of the segment,  $B$  and  $C$ .

Professor: Right. Tell me, can the side lengths be any three numbers we choose?

Ernest: As long as they're positive, why not?

Professor: Can you draw a triangle with sides of length 2, 3, and 7?

Katie: No way, the 7 is too large.

Sarah: I see what the problem is. The triangle inequality says that the sum of two sides—in this case  $2 + 3 = 5$ —must always be larger than the third side.

Ernest: Duh, that's why it's called the *triangle* inequality.

Chris: It's kind of strange that you can use any three pairs of points for the vertices—well, as long as they are not collinear—but you can't just use any three values for the lengths of the sides.

Professor: Interesting observation. The restrictions imposed by the triangle inequality, that is, that the sum of any two sides must always exceed the third, seems way more restrictive. It makes you think that side length is not the optimal way to describe a triangle. Can you think of any other way to generate triangles?

Sarah: If you draw three random lines, then the region inside them is a triangle.

Professor: Always?

Sarah: I guess it doesn't work if two of the lines are parallel. But then again, who would expect that random lines would turn out to be parallel?

Katie: They couldn't all meet in one common point, either, or else the triangle would be a single point.

Professor: Exactly. Now, if I have three positive real numbers whose values do not violate the triangle inequality, that is, the sum of any two of them is greater than the third, then can I always draw a triangle whose lengths are these three numbers?

Ernest: I can't see any reason why you can't, but I can't see any way to prove why you always can.

Professor: Let me ask a slightly different question. Suppose I wanted a triangle  $\triangle ABC$  where the measure of angle  $\angle B$  is, say 67, and the lengths  $AB$  and  $BC$  are 7 and 29. Can you always draw such a triangle?

Chris: Well, you could use the Protractor Axiom to construct the 67 degree angle.

Joshua: And then you could use the Ruler Axiom to make the sides the required lengths.

Ernest: Thank God for Axioms.

Professor: Here's my final question to think about for the next class. Suppose that Joshua and Sarah each make a triangle whose sides have length 7 and 29, where the in-between angle measures 67. Will their two triangles look exactly the same?

#### 4.8 Exercises

1. Draw a triangle  $\triangle ABC$ . Now draw a triangle  $CDE$ , where  $E$  lies on  $\overline{AC}$  and  $C - B - D$ . Which angle of  $\triangle CDE$  is congruent to an angle of  $\triangle ABC$ .
2. Try do this problem without drawing a picture. Fill-in the blanks. In triangle  $\triangle PQR$ ,
  - (a) segment  $\overline{RQ}$  is opposite \_\_\_\_\_.
  - (b) angle  $\angle PRQ$  is opposite \_\_\_\_\_.
  - (c) segment  $\overline{QP}$  is opposite \_\_\_\_\_.

### 9. Congruence

Professor: Okay, class, today we are discussing congruence. What does it mean to say that two geometric figures are congruent?

Katie: It means they have the same size and shape.

Professor: What does this mean, precisely.

Joshua: That you could trace the first one and then lay the tracing over the second figure and the two would perfectly coincide.

Professor: Very good. In mathematical terms, laying a tracing of one figure over another can be thought of as subjecting it to a rigid motion. You can translate it, rotate it, reflect it, but you cannot bend, stretch, or shrink it.

Sarah: Kind of the opposite of the rules in topology.

Professor: Good observation, Sarah. I do have a specific figure in mind. What does it mean to say that two segments  $\overline{AB}$  and  $\overline{XY}$  are congruent?

Ernest: They have the same length?

Professor: Right. And what does it mean to say that two angles  $\angle A$  and  $\angle X$  are congruent?

Hedda: They have the same angle measure.

Professor: Correct. Now what does it mean to say that two triangles  $\triangle ABC$  and  $\triangle XYZ$  are congruent?

Joshua: That's a lot harder. It means that corresponding sides are congruent and corresponding angles are congruent.

Ernest: C. P. C. T.

Professor: What does that stand for?

Ernest: We learned it in high school. It means congruent parts of corresponding triangles.

Hedda: Don't you mean corresponding parts of congruent triangles?

Ernest: Whatever.

Professor: Getting back to triangles  $\triangle ABC$  and  $\triangle XYZ$ , how do you know what side of  $\triangle XYZ$  corresponds to, say, side  $\overline{AC}$  of  $\triangle ABC$ ?

Joshua: You pair the vertices in the order you write them. Since you wrote the vertices of the first triangle in the order  $A, B, C$  and the vertices of the second triangle in the order

$X, Y, Z$ , the pairing is  $A$  with  $X$ ,  $B$  with  $Y$ , and  $C$  with  $Z$ . So side  $\overline{AC}$  of the first triangle corresponds to side  $\overline{XZ}$  of the second.

Professor: Are you suggesting that in order to show that  $\triangle ABC$  is congruent to  $\triangle XYZ$  we must show

$$\begin{array}{lll} AB = XY & & m\angle A = m\angle X \\ BC = YZ & \text{and} & m\angle B = m\angle Y \\ CA = ZX & & m\angle C = m\angle Z \end{array}$$

Ernest: That's a lotta work, Professor.

Katie: There's gotta be an easier way.

Professor: And there is. Can any one tell me what it is?

Katie: Something about Side – Angle – Side.

Ernest: I've heard of that, but I never really understood it.

Joshua: It means that if you match up two sides

$$AB = XY \quad \text{and} \quad AC = XZ$$

and the angle in between these sides

$$m\angle A = m\angle X,$$

then you know the two triangles are congruent, without having to check the other side or the other two angles.

Hedda: That saves half the work.

Professor: How do you know this always works?

Ernest: It says so in our book.

Professor: Why does the book say so?

Hedda: Because it's a true mathematical fact.

Professor: As a fact, is it a theorem or is it an axiom?

Ernest: Huh?

Professor: I mean, can you prove the Side–Angle–Side Statement using the other axioms of geometry or must we accept it as an axiom?

Katie: It seems a lot more complicated than our other axioms.

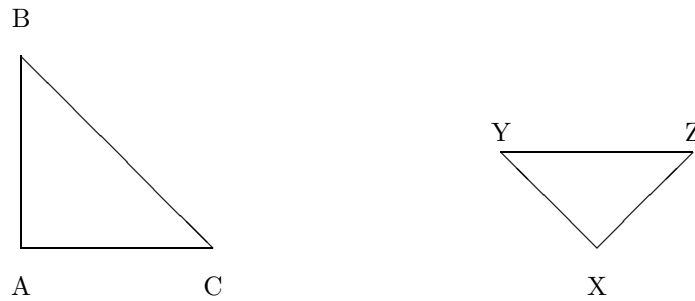
Professor: Yes, it does. As it turns out, however, it is an axiom of geometry. You can't really prove it from the other axioms.

Sarah: How can you be sure? Maybe there's a proof that is really hard and you just haven't thought of it.

Professor: Let me explain. Suppose you could prove Side–Angle–Side from the other axioms. The  $x$ - $y$  plane with the taxi-cab distance turns out to satisfy all these axioms, so it would have to obey Side–Angle–Side. But it doesn't.

Ernest: No lie?

Professor: Really. You can convince yourself. Here's a hint. Study the following two triangles:



The coordinates are:

$$A = (0, 0) \quad B = (0, 1) \quad C = (1, 0)$$

and

$$X = (0, 0) \quad Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \quad Z = \left(\frac{1}{2}, \frac{1}{2}\right)$$

**Do–It–Now Exercise.** Using the taxicab distance function, show that

(a)  $AB = XY$

(b)  $AC = XZ$

(c)  $m\angle A = m\angle X$

Does  $BC = YZ$ ?  $\square$

The Professor's example demonstrates that Side–Angle–Side is actually an axiom—not a theorem—of geometry.

**Side–Angle–Side (SAS) Axiom.** Given a correspondence between triangles  $\triangle ABC$  and  $\triangle XYZ$ . If (i)  $AB = XY$ , (ii)  $AC = XZ$ , and (iii)  $m\angle A = m\angle X$ , then  $\triangle ABC$  is congruent to  $\triangle XYZ$ .

In words, if two sides of one triangle are congruent to the corresponding sides of a second triangle, and if the angles between these two sides have the same measure, then the two triangles are congruent.

Finally, a word about notation. The traditional symbol for “is congruent to” is “ $\cong$ .” The statement “angle  $A$  is congruent to angle  $B$ ” can be written as “ $\angle A \cong \angle B$ ”, or equivalently, as the equation “ $m\angle A = m\angle B$ .” The statement  $\angle A = \angle B$ , on the other hand, asserts that the two angles are identical, not just in measure, but in their location.

#### 4.9 Exercises

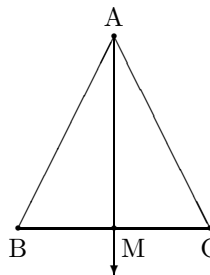
1. Does Side–Angle–Side hold on the surface of a football? Measure the hypotenuse of a right triangle drawn in the middle, with one leg along the seam, and the hypotenuse of a right triangle drawn at the end of the football.

2. Given:  $\triangle ABC$

$$AB = AC$$

the angle bisector of  $\angle A$  crosses  $\overline{BC}$  at point  $M$

Prove:  $M$  is the midpoint of  $\overline{BC}$ .



3. [Construct the Segment] Given a proper angle  $\angle BAC$  and a point  $P$  in the interior of this angle. (This means that  $P$  and  $C$  are on the same side of  $\overleftrightarrow{AB}$  and that  $P$  and  $B$  are on the same side of  $\overleftrightarrow{AC}$ .) Show how to use our axioms to obtain points  $X$  and  $Y$  such that

- (i)  $X$  is on  $\overrightarrow{AB}$
- (ii)  $Y$  is on  $\overrightarrow{AC}$
- (iii)  $P$  is the midpoint of  $\overline{XY}$

[Hint: Use the Ruler Axiom to locate a point  $Q$  on  $\overrightarrow{AP}$  such that  $AQ = 2AP$ .]

## 10. Isosceles Triangles and Triangle Congruence Theorems

An easy and instructive application of Side–Angle–Side is in the proof of the Isosceles Triangle Theorem.

**Definition 4.1.** An **isosceles** triangle is a triangle which has at least two congruent sides. An **equilateral** triangle has all three sides congruent. A triangle in which all three lengths are different numbers is called **scalene**.

**Definition 4.2.** A **right** triangle is a triangle, one of whose angles is a right angle. The side opposite the right angle is called the **hypotenuse** of the right triangle; the other two sides are called the **legs** of the triangle.

A well known fact from high school geometry is

**Theorem 4.1.** [Isosceles triangle theorem] Given  $\triangle ABC$ . If  $AB = AC$ , then  $m\angle B = m\angle C$ .

This theorem asserts that the corresponding angles of an isosceles triangle are congruent.

To prove this theorem, we set up the following 1–1 correspondence between  $\triangle ABC$  and  $\triangle ACB$ , the same triangle with the vertices  $B$  and  $C$  ordered differently. By hypothesis

$$AB = AC \quad \text{and} \quad AC = AB.$$

Moverover, by the symmetry axiom of angle measurement,

$$m\angle BAC = m\angle CAB.$$

By Side–Angle–Side,  $\triangle ABC$  is congruent to  $\triangle ACB$ . It follows that corresponding angles are congruent. Hence

$$m\angle ABC = m\angle ACB. \quad \square$$

What about the converse of this theorem? If we know that  $\angle B$  is congruent to  $\angle C$  does it follow that  $AB = AC$ , that is, that the triangle is isosceles? The answer is “yes” but the proof requires a different method for proving triangle congruence than Side–Angle–Side.

We discuss four other methods for establishing triangle congruence.

**Side–Side–Side [SSS]** Given  $\triangle ABC$  and  $\triangle XYZ$ . If  $AB = XY$ ,  $AC = XZ$ , and  $BC = YZ$ , then the two triangles are congruent.

SSS is a true statement. It is not an axiom, however, but can be proved using Side–Angle–Side [or SAS], although we will not do so in these notes.

**Angle–Side–Angle [ASA]** Given  $\triangle ABC$  and  $\triangle XYZ$ . If  $AB = XY$ ,  $m\angle A = m\angle X$ , and  $m\angle B = m\angle Y$ , then the two triangles are congruent.

**Angle–Angle–Side** Given  $\triangle ABC$  and  $\triangle XYZ$ . If  $m\angle A = m\angle X$ ,  $m\angle B = m\angle Y$ , and  $BC = YZ$ , then the two triangles are congruent.

ASA and AAS are both true statements which can be proved using SAS.

Now we come to the tricky one:

**Side–Side–Angle [SSA]** (*Warning:* Reversing these three letters leads to snickers among seven graders.) Given  $\triangle ABC$  and  $\triangle XYZ$ . If  $AB = XY$ ,  $BC = YZ$ , and  $m\angle C = m\angle Z$ , then are the two triangles congruent?

It turns out that SSA is *not* a valid method for establishing triangle congruence. SSA is “almost” true, however, as the following examples show.

**Example 4.1.** Suppose we specify  $\angle C$  and the two lengths  $BC$  and  $AB$ .

- (1) If  $m\angle C = 45$ ,  $BC = 2$ , and  $AB = 1.75$ , then two different triangles can be drawn matching these conditions. One way  $\angle A$  is obtuse, the other way it is acute.
- (2) No triangle can be drawn with the conditions  $m\angle C = 45$ ,  $BC = 2$ , and  $AB = 1$ . The specified length  $AB = 1$  is not long enough for side  $\overline{AB}$  to reach  $\overleftarrow{AC}$ .
- (3) Exactly one triangle can be drawn with the conditions  $m\angle C = 90$ ,  $BC = 2$ , and  $AB = 3$ . In general SSA is a valid method for triangle congruence if the angle is a right angle. This fact is important enough to have its own name.

**Hypotenuse–Leg [HL]** If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of a second right triangle, then the two triangles are congruent.

We shall feel free to employ all the methods—SAS, SSS, ASA, AAS, and HL—listed in this section to establish triangle congruence. The important thing for you to realize is that only Side–Angle–Side is an axiom; the others can be proved, though we shall not do them all. As an illustration, we outline a proof of Angle–Side–Angle.

Given  $\triangle ABC$  and  $\triangle XYZ$ . Suppose  $\angle A \cong \angle X$ ,  $\angle B \cong \angle Y$ , and  $AB = XY$ . We wish to prove that  $\triangle ABC \cong \triangle XYZ$ . Compare  $BC$  and  $YZ$ . If  $BC = YZ$ , then the two triangles

are congruent by SAS, and we're done. So assume  $BC \neq YZ$ . Without loss of generality, we may assume that  $BC > YZ$ .

*First Time-out.* What's all this "without loss of generality" stuff? Math books are always saying annoying things like this and students are generally thinking "whatever." How can you just assume that  $BC > YZ$ ? If  $BC \neq YZ$ , isn't the inequality  $BC < YZ$  just as likely to occur? The answer is that if  $BC$  is shorter than  $YZ$ , then we could just switch the letters  $A, B, C$  and  $X, Y, Z$ , so that we would have  $BC > YZ$ . Can you just switch letters like this? Sure, why not? We started with two general triangles—one we called  $\triangle ABC$ , the other we labelled  $\triangle XYZ$ . Reversing these names from the outset would not have changed the problem. The statements  $\angle A \cong \angle X$  and  $\angle B \cong \angle Y$  still hold if we switch  $A$  with  $X$  and  $B$  with  $Y$ ; the same goes for the equation  $AB = XY$ . Had there been something different about the two triangles, for instance, if we assumed that one of the triangles was isosceles, but not necessarily the other triangle, then we could not be so free in switching letters.

Back to the proof. Use the Ruler Axiom to construct a point  $D$  on  $\overrightarrow{BC}$  such that  $BD = YZ$ . The point  $D$  lies inside  $\overline{BC}$ , otherwise  $D$  would satisfy the betweenness relation  $B - C - D$ , implying that  $BC < BD = YZ$ , which is false. If you're thinking "couldn't  $D = C$ ?", forget it:  $D = C$  leads to  $BC = BD = YZ$ , another false statement. The only choice left is  $B - D - C$ , proving that  $D$  lies inside  $\overline{BC}$ , as claimed. By SAS,  $\triangle ABD$  is congruent to  $\triangle XYZ$ . Matching corresponding parts of congruent triangles, we get the angle congruence  $\angle BAD \cong \angle YXZ$ . Switching gears, we argue that since  $B - D - C$ , it surely follows that  $\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AC}$ .

*Second Time-out.* Not so fast. How do we know that the ordering of the rays  $\overrightarrow{AB}$ ,  $\overrightarrow{AD}$ , and  $\overrightarrow{AC}$  preserves the ordering of the points  $B$ ,  $D$ , and  $C$ ? The answer is, "We don't. We really need another axiom to guarantee this.

**Order Compatability Axiom.** Given a triangle  $\triangle ABC$  and a point  $D$  on line  $\overleftrightarrow{BC}$ . The betweenness relation  $B - D - C$  holds if and only if  $\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AC}$  is true.

If you spotted this problem in the proof, then you should consider a career in either law or mathematics. By the way, if you are thinking to yourself "couldn't you just see the correct betweenness relation from the diagram?", the answer is that diagrams can be misleading. Section 11 is all about what happens when we rely too much on diagrams to determine betweenness relations. In general, we want our proofs to rely solely on axioms, already established theorems, and logic.

Back to the proof. The betweenness relation  $\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AC}$  implies that  $m\angle BAD < m\angle BAC$ . The first angle  $\angle BAD$  is congruent to  $\angle YXZ$ , as noted above. But the second angle  $\angle BAC$  is also congruent to  $\angle YXZ$ , by hypothesis. So we have proved that  $m\angle YXZ = m\angle BAD < m\angle BAC = m\angle YXZ$ . This obvious contradiction originated from the assumption that  $BC \neq YZ$ . We must have been wrong to make this assumption, and in fact,  $BC = YZ$ , and consequently,  $\triangle ABC \cong \triangle XYZ$ .

#### 4.10 Exercises

1. Draw a picture illustrating each of the three examples of SSA listed above.
2. Use ASA to prove the converse of the isosceles triangle theorem, that is, that in  $\triangle ABC$ , if  $m\angle B = m\angle C$ , then  $AB = AC$ .
3. Prove: If the hypotenuse and one acute angle of a right triangle are congruent to the hypotenuse and one acute angle of a second right triangle, then the two triangles are congruent.
4. Use the Order Compatibility Axiom to prove: given triangle  $\triangle ABC$  and point  $D$  on ray  $\overrightarrow{BC}$ , we have that  $BD > BC$  if and only if  $m\angle BAD > m\angle BAC$ . You can test this theorem with a pair of scissors and a ruler.

## 11. Perpendiculars

**Definition 4.3.** Two angles  $\angle A$  and  $\angle B$  are called **supplementary** if and only if

$$m\angle A + m\angle B = 180.$$

**Theorem 4.2** (Supplementary Angle Theorem). Let  $h$  and  $k$  be opposite rays with a common endpoint  $A$ . Suppose  $r$  is a ray with endpoint  $A$  other than  $h$  or  $k$ . Then  $h - r - k$ . Consequently, the angle formed by the rays  $h$  and  $r$  and the angle formed by the rays  $r$  and  $k$  are supplementary.

**PROOF.** Let  $\ell$  be the line  $h \cup k$ . Since  $h$ ,  $r$ , and  $k$  lie on the same halfplane determined by the line  $\ell$ , the Betweenness Axiom for Rays guarantees that a betweenness relation

$$h - r - k, \quad h - k - r, \quad \text{or} \quad r - h - k$$

holds. Since  $m(h, k) = 180$  by Angle Measure Axiom 3, the only possible betweenness relation among the three listed above is

$$h - r - k.$$

(The second choice  $h - k - r$  implies  $m(h, r) = m(h, k) + m(k, r) = 180 + m(k, r) > 180$ , a contradiction; similarly, the third choice  $r - h - k$  implies  $m(r, k) > 180$ .) By definition of betweenness of rays, we have  $m(h, r) + m(r, k) = m(h, k) = 180$ . Hence the angles formed by  $h, r$  and  $r, k$  are supplementary.  $\square$

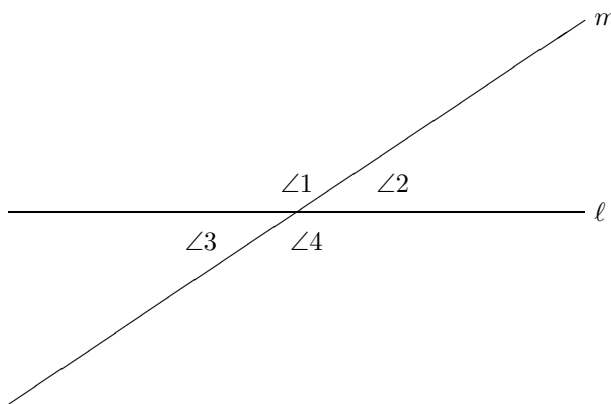
An immediate consequence of the definition of supplementary angles is

**Theorem 4.3** (Congruent Supplementary Angle Theorem). Let  $\angle A$  and  $\angle B$  be congruent supplementary angles. Then  $\angle A$  and  $\angle B$  are both right angles.

**Do-It-Now Exercise.** Find a (really) easy way to prove this theorem.

**Definition 4.4.** Two lines  $\ell$  and  $m$  are called **parallel** if and only if they do not intersect.

Two non-parallel lines meet in exactly one point, because of Incidence Axiom 3 which asserts that exactly one line passes through two distinct points. At the point of intersection of non-parallel lines  $\ell$  and  $m$ , we can form four angles:



**Definition 4.5.** In the above diagram, the pair of angles  $\angle 1$  and  $\angle 4$  as well as the pair  $\angle 2$  and  $\angle 3$  are called **vertical** angles.

By the Supplementary Angle Theorem,

$$m\angle 1 + m\angle 2 = 180$$

and

$$m\angle 1 + m\angle 3 = 180.$$

It follows that

$$m\angle 2 = m\angle 3.$$

A similar argument proves that

$$m\angle 1 = m\angle 4.$$

This proves

**Theorem 4.4** (Vertical Angle Theorem).

When two lines cross, their vertical angles are congruent.

**Definition 4.6.** We say that two non-parallel lines  $\ell$  and  $m$  are **perpendicular** if in the intersection diagram  $m\angle 1 = 90$ .

It follows from the Supplementary and Vertical Angles Theorems that when  $\ell$  and  $m$  are perpendicular, then all four angles are right angles. The Protractor Axiom guarantees that we can construct a perpendicular line at any point  $A$  on a given line. It is worth stating this as a theorem.

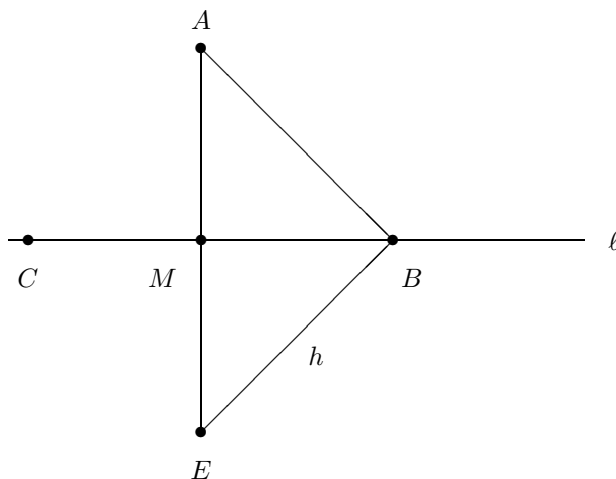
**Theorem 4.5.** Given any point  $A$  on a line  $\ell$ .

There is a line through  $A$  that is perpendicular to  $\ell$ .

Now suppose we are given a line  $\ell$  and a point  $A$  not on  $\ell$ . Is it always the case that we can construct a line through  $A$  that is perpendicular to  $\ell$ ? The answer is “yes” and here’s how.

Pick any point  $B$  on line  $\ell$ . Now pick a second point  $C$  on  $\ell$  so that  $m\angle ABC \leq 90$ . How do we know we can always find such a point  $C$ ? We begin by choosing  $D$  to be any point other than  $B$  on  $\ell$ . If  $m\angle ABD \leq 90$ , fine; take  $C = D$ . If  $m\angle ABD > 90$ , then choose a point  $C$  on  $\ell$  so that  $\overrightarrow{BC}$  and  $\overrightarrow{BD}$  are opposite rays. Then  $\angle ABC$  and  $\angle ABD$  are supplementary angles. Hence,  $m\angle ABC \leq 90$ .

Now if  $m\angle ABC$  is exactly 90, stop, we’re done; line  $\overleftrightarrow{AB}$  is perpendicular to line  $\overleftrightarrow{BC}$ . We are left with the case where  $m\angle ABC < 90$ . On the side of line  $\ell$  which does not contain  $A$  construct a ray  $h$  with endpoint  $B$  such that the angle measure between  $\overrightarrow{BC}$  and ray  $h$  is precisely  $m\angle ABC$ . Using the Ruler Axiom, construct a point  $E$  on ray  $h$  so that  $BE = BA$ .



Observe that  $A$  and  $E$  are on opposite sides of  $\ell$ , so the line segment  $\overline{AE}$  must cross line  $\ell$  at a point  $M$ . Examine the two triangles  $\triangle ABM$  and  $\triangle EBM$ . By construction

$$m\angle ABM = m\angle ABC = m\angle EBC = m\angle EBM.$$

Furthermore,

$$AB = EB$$

and finally, the two triangles share the common side  $\overline{BM}$ . By axiom Side–Angle–Side,

$$\triangle ABM \cong \triangle EBM.$$

This means

$$\angle AMB \cong \angle EMB.$$

Since  $A$ ,  $M$ , and  $E$  are collinear, the angles  $\angle AMB$  and  $\angle EMB$  are supplementary angles. But we know that congruent, supplementary angles must be right angles. We have found our perpendicular line:  $\overleftrightarrow{AE}$  is perpendicular to  $\ell$ .

We have just proven the following statement:

**Theorem 4.6.** Given a line  $\ell$  and a point  $A$  not on  $\ell$ .

Then there is a line  $t$  through  $A$  and perpendicular to  $\ell$ .

The line  $t$  guaranteed by Theorem 4.6 crosses the original line  $\ell$  at a point  $B$ . We define the **distance from  $A$  to line  $\ell$**  to be the distance  $AB$ .

#### 4.11 Exercises

1. Suppose two lines  $\ell$  and  $m$  intersect at point  $A$ . We say that the point  $P$  is **equidistant** from lines  $\ell$  and  $m$  if the (perpendicular) distance from  $A$  to  $\ell$  equals the (perpendicular) distance from  $A$  to  $m$ . Describe the sets of all points  $P$  which are equidistant from  $\ell$  and  $m$ .

2. Given:  $\overleftrightarrow{AB}$  is perpendicular to  $\overleftrightarrow{AD}$   
 $\overleftrightarrow{BC}$  is perpendicular to  $\overleftrightarrow{CD}$ .

$$AB = BC$$

Prove that  $AD = DC$ .

3. Given  $\triangle ABC$ . Prove:  $m\angle B + m\angle C < 180$ .

Outline of the proof:

Let  $M$  be the midpoint of  $\overline{AB}$ .

Let  $E$  be a point on the ray  $\overrightarrow{CM}$  such that  $CE = 2CM$ .

Show that  $\triangle AMC \cong \triangle BME$ .

Show that  $\overline{BC} = \overline{BM} + \overline{BE}$ . Hint: Use the Order Compatibility Axiom.

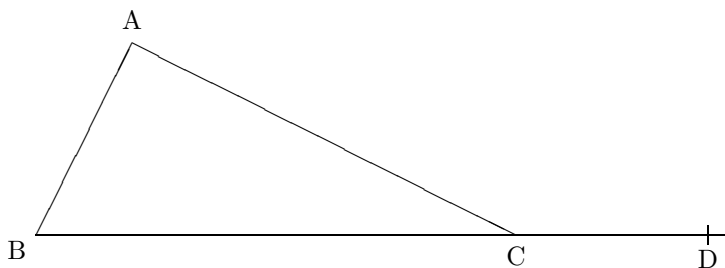
Show that  $m\angle A + m\angle B = m\angle CBE$ .

Since  $\angle CBE$  is a proper angle, conclude that  $m\angle A + m\angle B < 180$ .

4. This exercise asks you to prove

**Theorem 4.7** (Exterior Angle Theorem). Given  $\triangle ABC$  and  $A - C - D$ . Then  $m\angle ABD > m\angle A$  and  $m\angle ABD > m\angle B$ .

Angle  $\angle ABD$  is called an **exterior angle** of triangle  $\triangle ABC$ . Angles  $\angle A$  and  $\angle B$  are called the **remote interior angles** corresponding to  $\angle ABD$ .



Hint: Use Exercise 3 and the Supplementary Angle Theorem.

### 12. Isosceles Triangle Paradox

Consider the following “proof” that all triangles are isosceles.

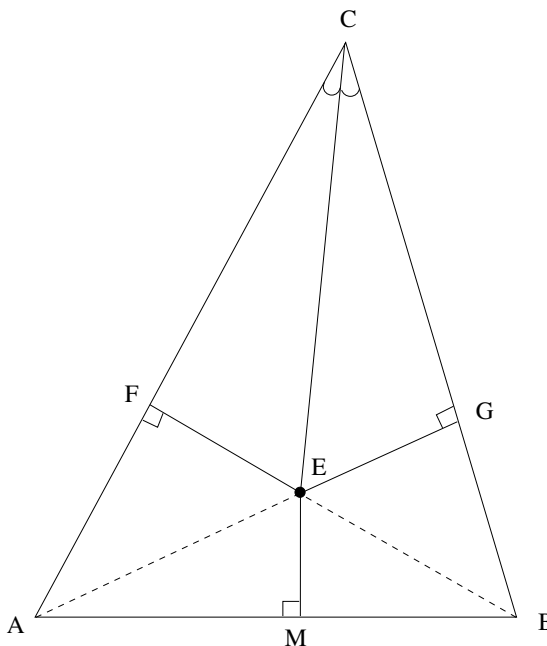


FIGURE 1. Triangle  $\triangle ABC$

1. Let  $M$  be the midpoint of  $\overline{AB}$ . Draw the bisector of  $\angle C$  and the perpendicular bisector of  $\overline{AB}$ . Let  $E$  be their point of intersection.
2. By construction,  $\overline{EM} \perp \overline{AB}$ . From point  $E$  drop perpendiculars  $\overline{EF}$  onto  $\overline{AC}$  and  $\overline{EG}$  onto  $\overline{BC}$ .
3.  $\triangle CFE \cong \triangle CGE$  by Angle–Angle–Side. Each is a right triangle with  $\overline{CE}$  as a hypotenuse and  $\angle FCE = \angle GCE$  since  $\overline{CE}$  bisects  $\angle C$ .
4.  $EF = EG$  and  $CF = CG$  [congruent parts of the congruent triangles  $\triangle CFE$  and  $\triangle CGE$ ].
5.  $\triangle AEM \cong \triangle BEM$  by Side–Angle–Side. Both are right triangles,  $AM = BM$  since  $M$  is the midpoint of  $\overline{AB}$ , and they share side  $\overline{EM}$ .
6.  $AE = BE$  [CPCT].

7.  $\triangle AEF \cong \triangle BEG$  by Hypotenuse–Leg. Both are right triangles,  $EF = EG$  from step 4, and  $AE = BE$  from step 6.

8.  $FA = GB$  [CPCT]

9. Combining steps 4 and 8, we have

$$CA = CF + FA = CG + GB = CB.$$

10. Since  $CA = CB$  it follows that  $\triangle ABC$  is isosceles.

Something is very wrong here! We have just “proved” a statement that is categorically false. Clearly not all triangles are isosceles. Try to find the flaw in the proof. It may help you to draw a triangle which is visually not isosceles, such as a 3–5–7 triangle. Construct the angle bisector of  $\angle C$  using a compass or protractor. Construct the perpendicular bisector of  $\overline{AB}$  using a compass or ruler. Where do these lines intersect? Do you see why the concept of betweenness is so important?