11. Parallels

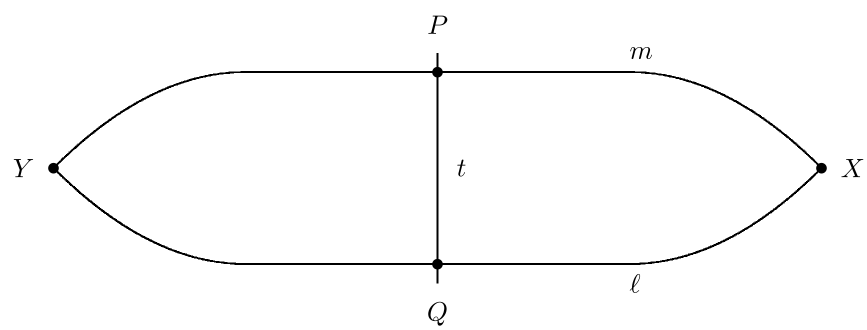
Given one rail of a railroad track, is there always a second rail whose (perpendicular) distance from the first rail is exactly the width across the tires of a train, so that the two rails never intersect? Of course we believe this is so. Let's state it as a theorem.

Theorem 4.1. Given a line ℓ and a point P not on ℓ.

$$\boxed{\text{Then there exists a line through } P \text{ parallel to } \ell.}$$

Okay, how do we go about proving Theorem 4.1? One idea is to use penpendiculars. Construct a line t through P and perpendicular to ℓ. Now construct a line m through P and perpendicular to t. Surely, lines m and ℓ must be parallel. Suppose you are driving on 8th Avenue in New York City. You turn right on 34th Street, go two blocks, and then turn right on 6th Avenue. Sixth Avenue must be parallel to 8th Avenue—and it is. But can we prove it?

Here goes. Suppose, to play Devil's Advocate, that m and ℓ are not parallel. Then they must meet at a point which we will call X (the "mysterious" point). Let Q be the point of intesection of line t and ℓ. Note that $\angle QPX$ and $\angle PQX$ are both right angles. Now let Y be a point on the opposite ray of $\overrightarrow{QX}$ so that $QY = QX$.



Consider the triangles $\triangle PQY$ and $\triangle PQX$. Since the angles $\angle PQY$ and $\angle PQX$ are supplementary and $m\angle PQX = 90$, it follows that $m\angle PQY = 90$. By construction $QY = QX$ and finally the two triangles share the side $\overline{PQ}$. By Side–Angle–Side,

$$\triangle PQY \cong \triangle PQX.$$

Consequently,

$$m\angle QPY = m\angle QPX = 90.$$

By angle addition,

$$m\angle YPX = m\angle YPQ + m\angle QPX = 90 + 90 = 180.$$

In other words, $\angle YPX$ is a straight angle, implying that Y, P, and X are collinear. We now have two different lines m and ℓ passing through the points X and Y. That ain't right! One of our basic axioms states that only one line passes through two distinct points. We were led to this contradiction by assuming that m and ℓ intersect; so, in fact, m and ℓ must be parallel. □

Now here comes the question that mystefied mathematicians for over two thousand years:

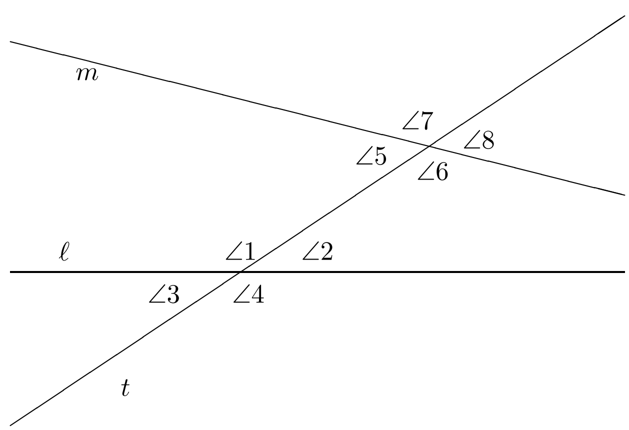
Is m the only parallel line through P?

It certainly seems plausible that any other line, different than the line constructed by perpendiculars in the previous theorem, must meet ℓ.

Theorem 4.2. [The Parallel Postulate] Given a line ℓ and a point P not on ℓ.

Then there exists a *unique* line through P parallel to ℓ.

A line t which crosses two given lines ℓ and m is called a **transversal**. In general, eight angles are formed:



Angles formed between the lines ℓ and m are called **interior** angles. In the above diagram, angles $\angle 1$, $\angle 2$, $\angle 5$, and $\angle 6$ are all interior angles. Note that interior angles come in pairs: $\angle 1$ and $\angle 5$ both lie on the left side of t, while $\angle 2$ and $\angle 6$ both lie on the right side of t.

Euclid's Fifth Postulate: In a transveral configuration, if the sum of the measure of the two interior angles on the same side of the transveral t is less than 180, then the lines ℓ and m will eventually intersect on this particular side of t.

This axiom seems self evident. Suppose $m\angle 2 + m\angle 6 < 180$. Then the two lines ℓ and m in the diagram above are "tilted" towards each other on the right side of t. If they are extended far enough they will eventually meet on the right side of t.

Euclid's Fifth Postulate shows that there cannot be more than one parallel line through a given point P. If the sum $m\angle 2 + m\angle 6$ is less than 180 then the two lines will meet on the side of t that contains $\angle 2$ and $\angle 6$; otherwise $m\angle 1 + m\angle 5 < 180$ and the two lines meet on the other side of t. Lines ℓ and m are parallel if and only if $m\angle 2 + m\angle 6$ is exactly 180. Interior angles which lie on opposite halfplanes of t are called **alternate interior** angles. In the diagram, the pair $\angle 1$ and $\angle 6$ are alternate interior angles. So are the pair $\angle 2$ and $\angle 5$.

Do–It–Now Exercise. Using Euclid's Fifth Postulate, prove

Theorem 4.3 (Alternate Interior Angles Theorem)**.** Given a configuration of two lines ℓ and m cut by a transversal t.

The lines ℓ and m are parallel if and only if alternate interior angles are congruent.

Do–It–Now Exercise. Show that Theorem 4.2, the Parallel Postulate, is an easy consequence of Theorem 4.3.

At this point it seems fair to ask: What is the status of Euclid's Fifth Postulate? Do we add it to our system of axioms or is it possible to prove it from the other axioms of geometry? For centuries mathematicians attempted to prove Euclid's Fifth Postulate (or equivalently, the Parallel Postulate) from the other axioms of Euclid. Finally it was realized that the search for the elusive proof was in vain. Geometries in which all the other axioms of Euclid hold except for the Parallel Postulate were discovered (invented?) which are as consistent as Euclidean geometry in which the Parallel Postulate is assumed to hold true.

Our plan is to stay tuned to the Euclid channel for now, accept Euclid's Fifth Postulate and see where it leads us. Later in Module 7 we will explore the "non-Euclidean" channels.

4.11 Exercises

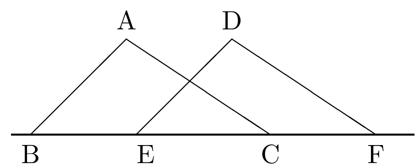
1. Given an isosceles triangle $\triangle ABC$, with $AB = AC$. Suppose ℓ is a line parallel to $\overleftrightarrow{AB}$ and that ℓ meets segment $\overline{AC}$ at point P and ℓ meets segment $\overline{BC}$ at point Q. Prove that $\triangle PCQ$ is isosceles.

2. Given: $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{DE}$

$AB = DE$

$BE = CF$

Prove: $\overleftrightarrow{AC}$ is parallel to $\overleftrightarrow{DF}$.



3. One way of constructing a line through a point P parallel to a given line ℓ is to use the Alternate Interior Angles Theorem. Here is another way.

Given a line ℓ and a point P not on ℓ.
Let A be any point on ℓ.
Locate the midpoint M of $\overline{AP}$.
Draw any line through M which meets ℓ at a point C other than A.
Find the point D on $\overrightarrow{CM}$ such that $CD = 2\,CM$.
Show that $\overleftrightarrow{BP}$ is parallel to line ℓ.

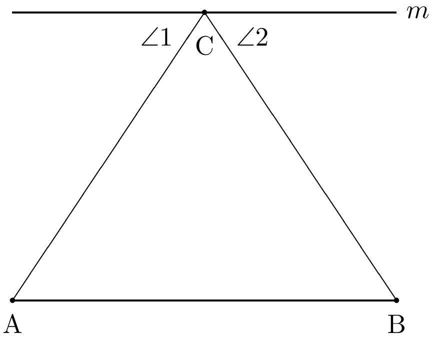
12. Angle Sums of Triangles

We are now ready to prove the most famous of all theorems in geometry.

Theorem 4.4. In $\triangle ABC$

$$\boxed{m\angle A + m\angle B + m\angle C = 180.}$$

The idea of the proof is to use the line m through A parallel to $\overleftrightarrow{BC}$.



By the Alternate Interior Angle Theorem,

$$m\angle A = m\angle 1$$

and

$$m\angle B = m\angle 2.$$

Thus

$$\begin{aligned} 180 &= m\angle 1 + m\angle C + m\angle 2 \\ &= m\angle A + m\angle C + m\angle B. \end{aligned}$$

We're done. □

Definition 4.1. Two angles $\angle A$ and $\angle B$ are said to be **complementary** if

$$\boxed{m\angle A + m\angle B = 90.}$$

Theorem 4.5. In triangle $\triangle ABC$, if $\angle C$ is a right angle, then $\angle A$ and $\angle B$ are complementarry.

4.12 Exercises

1. If m$\angle A = 40$ and m$\angle B = 70$, what is m$\angle C$ of $\triangle ABC$?

2. If m$\angle A =$ 2m$\angle B$ and m$\angle C$ of $\triangle ABC$ is 120, what is m$\angle A$ and m$\angle B$?

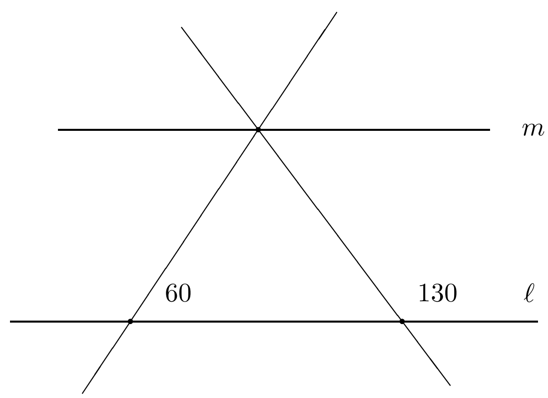
3. If one angle of an isosceles triangle has measure 45, what are the measures of the other two angles?

4. What is the measure of an angle of an equilateral triangle?

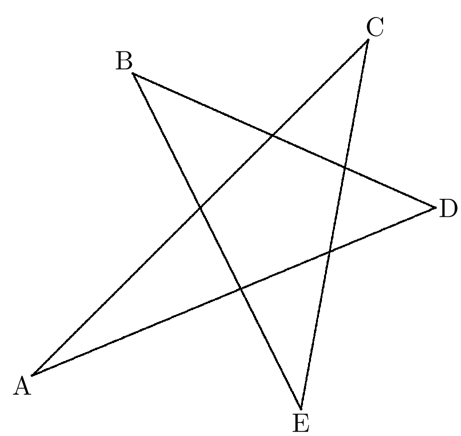
5. Prove that the measure of the short leg of a 30–60–90 triangle is half the measure of the hypotenuse.

6. Explain why the triangle congruence Angle–Angle–Side really just follows from Angle–Side–Angle.

7. Given that lines ℓ and m are parallel. Find the measures of the other 12 angles in the following diagram:



8. [The Star Problem] Given a five pointed star:



Prove that $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E = 180.$

9. Given: $B - C - D$

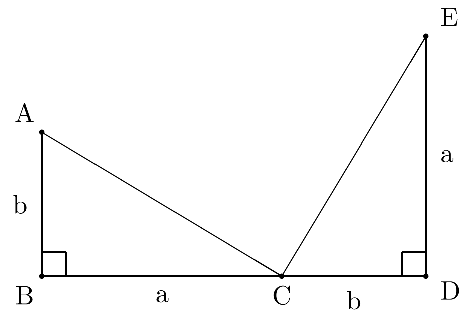
$\angle ABC$ is a right angle

$\angle CDE$ is a right angle

$AB = CD$

$BC = DE$

Prove: $\angle ACE$ is a right angle.



In the Euclidean model of the $x{-}y$ plane, suppose the points $A - E$ are given by

$$A = (0, b), \;\; B = (0, 0), \;\; C = (a, 0), \;\; D = (a + b, 0), \;\; E = (a + b, a)$$

(a) What is the slope of line $\overleftrightarrow{AC}$?

(b) What is the slope of line $\overleftrightarrow{CE}$?

(c) What conclusion can you make about the slopes of perpendicular lines?