

MODULE 5

Topics in Geometry

Euclid alone has looked on beauty bare. – Edna St. Vincent Millay

1. Polygons

Definition 5.1. A **polygon** is a union of a finite number of line segments satisfying the following conditions:

- (i) each end point is an end point of exactly two segments;
- (ii) no two segments intersect except at an end point; and
- (iii) two segments with a common end point are not collinear.

We specify a polygon by listing its vertices in order, such as $ABCD$. This polygon is called a **quadrilateral**, meaning it has four sides \overline{AB} , \overline{BC} , \overline{CD} , and \overline{DA} .

Specials types of quadrilaterals:

parallelogram: opposite sides are parallel.

rectangle: its angles are all right angles.

square: a rectangle with two adjacent sides congruent.

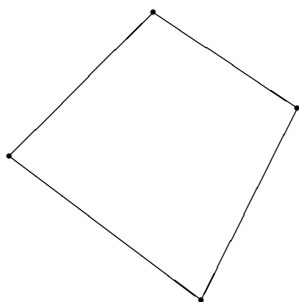
rhombus: all its sides are congruent.

trapezoid: just two of its sides are parallel. The parallel sides are called the **bases**; the nonparallel sides are called the **legs**.

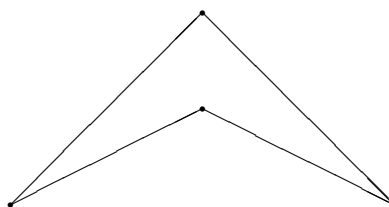
isosceles trapezoid: a trapezoid whose legs are congruent.

A polygon is **convex** if the line segment joining any two points inside the polygon lies entirely in the polygon. Another way to think of this is to consider driving nails at the

vertices of the polygon. If a rubber band stretched over the nails touches all the nails, then the polygon is convex.



convex



not convex

5.1 Exercises

1. Explain why the quadrilateral $ABCD$ can also be described as $BCDA$ or $DCBA$, but not $ACBD$.
2. Prove that each rectangle is a parallelogram.
3. Prove that opposite sides of a parallelogram are congruent.
4. Let M and N be the two midpoints of the legs (nonparallel sides) of a trapezoid. Prove that \overline{MN} is parallel to each base (parallel side) of the trapezoid.
5. Exercise 2 states that all rectangles are parallelograms. It is clear from the definition that all squares are rectangles. List all such relationships between any two types of quadrilaterals listed above.
6. Are all triangles (3-sided polygons) convex? Are all quadrilaterals (4 sided polygons) convex? Are all pentagons (5-sided polygons) convex?
7. The following theorem is very useful to carpenters who want to make sure that the walls of a room are square. Show that a parallelogram $ABCD$ is a rectangle if and only if $AC = BD$.

2. Area

2.1. The Three Properties of Area.

Professor: Today's topic is area. Who can tell me what the area of a region is?

Paul: It's the space occupied by the region.

Professor: Do you mean the set of points inside the region.

Paul: I guess so.

Professor: Let me ask a different question. What is the area of a rectangle whose sides have length 2 and 3?

Ernest: That's easy—it's six.

Professor: Right. Now Paul, if you agree with Ernest, then do you think the number six is the same as a set of points inside a rectangle?

Paul: Of course not! I mean that area is the space taken up by the rectangle.

Professor: So you think the number 6 is some kind of space?

Paul: It's not the space itself.

Professor: Can somebody help her out?

Sam: It's the measure of the amount of space inside the region.

Professor: Very good. How do we know how to measure this "space" inside a region?

Ernest: It depends on the region.

Professor: Okay, suppose the region is a triangle. How do we measure its area?

Beth: Area equals one half base times height.

Professor: What is base?

Beth: The length of the bottom of the triangle.

Professor: And what is the height?

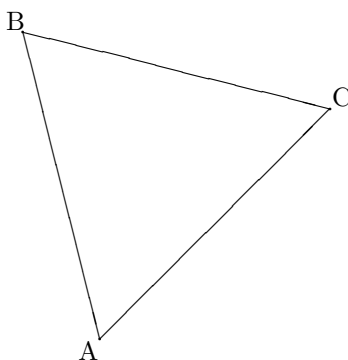
Beth: The length of the perpendicular draw from the third vertex to the base.

Paul: Isn't the perpendicular from a vertex to a side called the *altitude*?

Professor: That's right.

Beth: Smarty-pants.

Professor: [Draws on the board] Here is a triangle



How do you know what side to use for the base? Suppose Beth uses \overline{AB} for the base, calculates the height from C to \overline{AB} and then uses the area formula

$$\boxed{\text{Area} = \frac{1}{2} \text{base} \times \text{height.}}$$

Now suppose Paul uses \overline{AC} for her base and uses the height from B to \overline{AC} . Since Paul uses different values for the base and the height, isn't it possible that they will get different answers for the area?

Hedda: I've studied with both of them and it's not only possible but almost certain that their answers will differ. *Class snickers.*

Professor: I'm assuming that neither one of them makes an arithmetic mistake.

Sam: I see what you're getting at, Professor. Theoretically it's possible to get three different answers for the area of the triangle because there are three different sides you can use for the base.

Ernest: Wouldn't you get the same answer no matter what side you use?

Hedda: You really need a theorem that says they're the same.

Professor: Could you state this theorem?

Hedda: Let's see. The theorem would say that the product of an altitude drawn from a vertex of any triangle to its opposite side times the length of that side is the same no matter what vertex we choose.

Professor: Could you prove this theorem?

Hedda: Stating a theorem and proving a theorem are two different things.

Professor: Maybe it would be a good homework assignment. But I'd like to think some more about area in general. What sort of properties do you think are basic for area? For example, what if two regions are congruent?

Matthew: Then they should have the same area.

Professor: Alright, let's make that Property 1. [Writes on the board:]

Property 1. *Congruent regions have the same area.*

Professor: What else?

Ernest: How about our triangle formula?

Professor: Okay, assuming that we prove Hedda's theorem. [Writes on the board:]

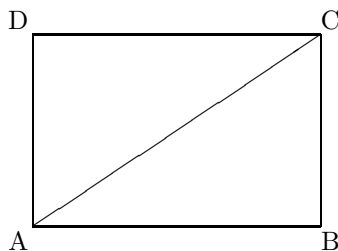
Property 2. *The area of a triangular region is one half the product of the length of any side times the length of the altitude drawn from the vertex opposite that side.*

Professor: What about other geometric shapes.

Beth: The area of a rectangle is length times width.

Professor: Should this be a new property or can we get it from Property 2.

Beth: Well, a rectangle can be split into two triangles:



By Property 2, each rectangle has area $\frac{1}{2}$ length \times width, so when we add them together we get double $\frac{1}{2}$ length \times width and the 2's cancel giving us the formula we want.

Professor: How do we know we can add them like this?

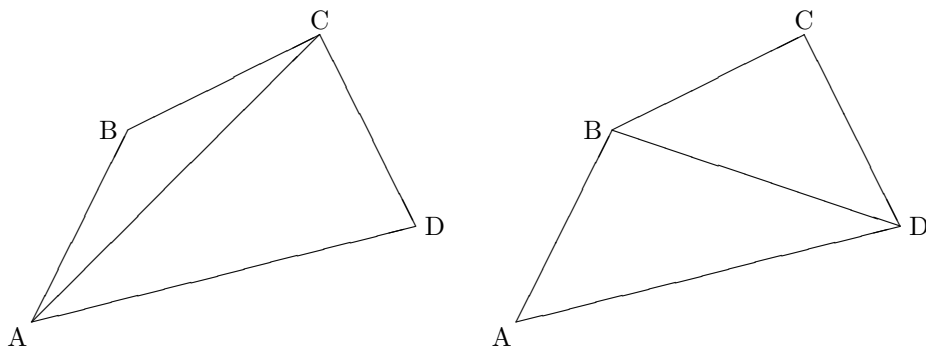
Ernest: Don't tell me, Professor, we need another property.

Professor: You got it. How about this? [Writes on the board:]

Property 3. *The area of the union of two regions whose intersection is contained in the boundary of each of them is the sum of the area of the two regions.*

Paul: Suppose you decide to compute the area inside a polygon by breaking it into triangles. There's more than one way to do this. How do you know that you will get the same number, no matter how you break it up?

Professor: That's a very good question. It's kind of like asking how you can be certain it doesn't matter what side you use for the base in computing the area of a triangle. Suppose we want to compute the area of a quadrilateral ABCD. We could split it up into two triangles using the diagonal segment \overline{AC} or we could split it up using the diagonal \overline{BD} like this



It turns out that no matter how a polygonal region is “cut up” using non-overlapping triangles, the sum of the areas of the triangular regions will be the same.

Ernest: Is this hard to prove?

Professor: Yes, it’s somewhat complicated.

Hedda: Isn’t it obvious?

Professor: Consider the same problem for finding the volume of three dimensional regions. In 1924 a mathematician named Banach and a logician named Tarski discovered a way to dissect a ball in 3-space into six pieces, so that when the pieces are subjected to rigid motions and reassembled the resulting region has twice the volume.

Sam: Cool.

Hedda: It doesn’t seem possible.

Professor: It doesn’t seem possible that you could have a set which has as many points as the real numbers, yet takes up no space on the number line, but such a set exists, namely, the Cantor set.

Hedda: Math is sure a lot stranger than I thought.

Ernest: So does this mean we can’t compute area by splitting up regions into triangles?

Professor: The dissection used in the Banach–Tarski Paradox is very elaborate. Rest assured that if you want to compute the area of polygonal regions in the plane, it is quite permissible to split them up into triangles in any way you choose.

5.2.1 Exercises

1. The area of a square is 144 square inches. Find the length of a side.
2. The length of \overline{AB} is 9. Describe the set of all points P such that the area of $\triangle ABP$ is 18.
3. One obvious problem with the area formula for a triangle is that it requires you to compute the length of an altitude. Such a calculation involves trig unless you are blessed to be working with a right triangle. A well-known two thousand year old formula, however, gives us the area of a triangle in terms of the lengths of the three sides. The formula, due

to Heron, is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the lengths of the sides and $s = \frac{1}{2}(a + b + c)$. Verify this formula when
 (a) $a = 3, b = 4, c = 5$; (b) $a = 5, b = 7, c = 9$.

4. Prove that the median of a triangle divides it into two triangles with the same area. A **median** is a line segment drawn from one vertex to the midpoint of the side opposite that vertex.

5. Using areas, give a geometric proof of the formula

$$(a + b)^2 = a^2 + 2ab + b^2$$

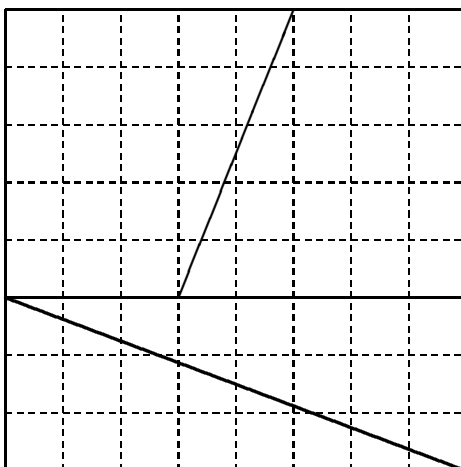
when a and b are positive.

6. Prove that the two diagonals of a parallelogram divide it into four triangles of the same area.

7. Prove that the area of a trapezoid is the length of the altitude between the bases times the average of the length of the two bases.

8. If the area of a trapezoid is 126, one base measures 12, and the altitude measures 9, what is the length of the other base?

9. Cut the following diagram along the three solid lines, to make four pieces. Reassemble these four pieces to form a 5 by 13 rectangle.



Since the original square has area 64 and the rectangle has area $5 \times 13 = 65$, where did the extra unit of area come from?

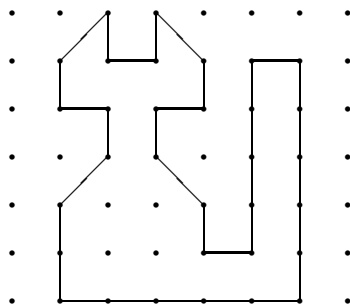
2.2. Pick's Theorem. It is not always easy to compute the area of a polygonal region. First you break the region into triangles, and then add up all the areas of the triangles. But computing the area of a triangle requires work. Unless you are lucky enough to be working with a right triangle, the area calculation may involve angle measurement, trigonometry to compute the altitudes, or at the very least, square roots to find the length of the sides.

Definition 5.2. A point P in \mathbb{R}^2 is called a **lattice point** if and only if the coordinates of P are both integers.

Now suppose the vertices of our polygon are all lattice points. Does this help in computing area? The answer, surprisingly, is that in this situation, we don't need trig, square roots, or any mathematics more complicated than *just counting points*. The secret is the following amazing result called Pick's Theorem:

Theorem 5.1 (Pick's Theorem). Give a polygonal region R whose vertices all have integer coordinates. Let B count the number of lattice points on the boundary of R and let I count the number of lattice point in the interior of R . Then the area A of R is

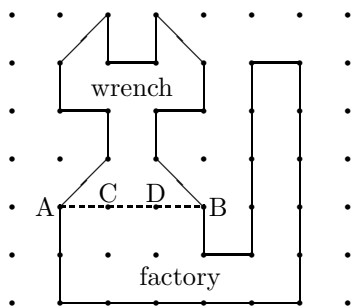
$$A = I + \frac{1}{2}B - 1.$$



By Pick's Theorem we can easily compute the area of the cat in the above picture. The number of interior points is $I_c = 4$ and the number of boundary points is $B_c = 32$, so the area is

$$A_c = I_c + \frac{1}{2}B_c - 1 = 4 + 16 - 1 = 19.$$

How would you go about proving Pick's Theorem? One way is to use Property 3, where smaller regions are joined to form larger regions. Suppose we break up our region into two pieces, whose intersection is a line segment whose endpoints have integer coordinates. For example, suppose we dissect the “cat” picture at the dotted line shown below, whose endpoints are A and B . The top piece looks like part of a “wrench” and the bottom piece looks like a flatroofed “factory” with a smokestack attached.



Now suppose that Pick's Theorem holds for both regions. Can we show that Pick's Theorem holds for the union of the two pieces? The wrench consists of $I_w = 0$ interior points and $B_w = 16$ boundary points. The factory consists of $I_f = 2$ interior points and $B_f = 22$ boundary points. By Pick's Theorem, the area of the wrench is

$$A_w = I_w + \frac{1}{2}B_w = 0 + \frac{1}{2}16 - 1 = 7$$

and the area of the factory is

$$A_f = I_f + \frac{1}{2}B_f = 2 + \frac{1}{2}22 - 1 = 12.$$

Notice that number of interior points of the cat is not the sum of the number of interior points of the two regions, that is, $I_w + I_f \neq I_c = 4$. The two interior points of the factory remain interior points when we glue the two regions to form the cat. But where do the other two interior points of the cat come from? Answer: they are the points C and D on the common edge \overline{AB} shared by the two smaller regions. Hence,

$$I_c = I_w + I_f + 2.$$

In the sum $B_w + B_f$, each of points, A , B , C , and D , are counted twice, once in B_w and again in B_f , whereas all other border points are counted just once. Since only the endpoints A and B remain border points when the regions are joined, it follows that

$$B_c = B_w + B_f - 4 - 2.$$

Be sure you understand this equation: we subtract 4 because the two points C and D were counted twice and they are no longer boundary points of the cat; we subtract 2 because the two endpoints A and B were counted twice, but they should only be counted once as boundary points of the cat. When we plug I_c and B_c into Pick's formula, we get

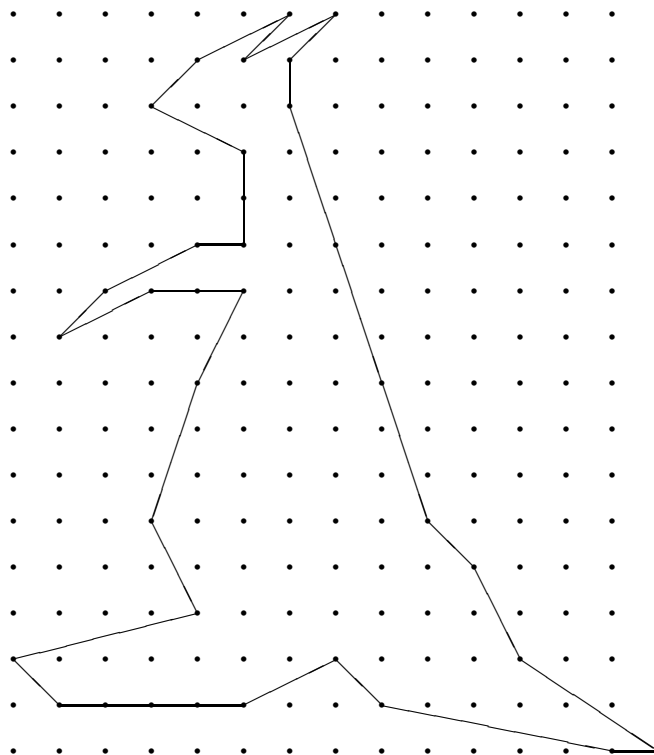
$$\begin{aligned} I_c + \frac{1}{2}B_c - 1 &= I_w + I_f + 2 + \frac{1}{2}(B_w + B_f - 6) - 1 \\ &= \left(I_w + \frac{1}{2}B_w - 1\right) + \left(I_f + \frac{1}{2}B_f - 1\right) = A_w + A_f, \end{aligned}$$

by Pick's Theorem. By Property 3, the area of the cat is the sum of these two areas, so we have just verified Pick's formula for the larger region.

You can break any polygonal region with integer coordinates into a finite number of triangles, whose vertices are lattice points. Proving Pick's Theorem for polygonal regions thus boils down to proving Pick's Theorem for triangles.

5.2.2 Exercises

1. Find the area of the kangaroo:



2. Prove that Pick's Theorem holds for any triangle $\triangle ABC$, whose vertices all have integer coordinates. Hint: first show that Pick's Theorem works for right triangles whose legs are parallel to the x and y axes; then enclose $\triangle ABC$ in a rectangle whose sides are parallel to the coordinate axes.
3. Construct your own lattice drawing on a sheet of graph paper.

3. The Pythagorean Theorem

Perhaps the most famous theorem in mathematics is the Pythagorean Theorem. The Scarecrow recites a version of the Pythagorean Theorem just after he gets a brain in the Wizard of Oz—unfortunately he gets it wrong. A former president of the United States actually published a proof of the Pythagorean Theorem using trapezoids. We examine this proof and others. We conclude with a group activity for extending the Pythagorean Theorem to higher dimensions.

As we begin, Professor Flappenjaw is preaching to his class.

Professor: Okay, class, tell me the most important theorem that you remember from your geometry class in high school.

Ernest: Area equals pi r squared.

Hedda: The sum of the three angles of a triangle is 180 degrees.

Paul: The Pythagorean Theorem.

Professor: What does the Pythagorean Theorem say?

Ernest: It can't talk, Professor, it's a theorem.

Professor: Enough of your wisecracks, Ernest. Can you state the Pythagorean Theorem?

Ernest: It's something about the square of the hippopotamus.

Professor: You mean hypotenuse! Tell me, Ernest, what is the hypotenuse?

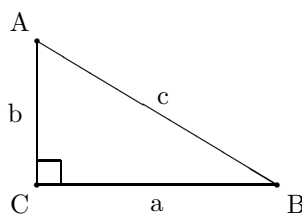
Ernest: The longest side of the triangle.

Professor: What if all sides have the same length, say, 5? Which side is the hypotenuse?

Ernest: You got me there.

Paul: Doesn't the triangle need to be a right triangle?

Professor: Right is right, I mean correct! And the side opposite the 90 degree angle must be the largest side, called the hypotenuse. Now what does the Pythagorean Theorem say about this hypotenuse? Perhaps it'll help if I draw a picture. [Draws on the board]



Sam: I remember. The Pythagorean Theorem states that

$$c^2 = a^2 + b^2.$$

Professor: Exactly. In the old Judy Garland movie *The Wizard of Oz* the first thing the Scarecrow says when his wish for a brain is granted is a mangled version of the Pythagorean Theorem.

Beth: What do you mean “mangled?”

Professor: He says “isosceles triangle” instead of “right triangle” and he forgets the squares on the two sides.

Matthew: Is that so that people watching the movie will realize that the Scarecrow doesn't have all the brains he thinks he has.

Professor: Frankly, I doubt if anyone ever spots the fact that the Scarecrow is spouting mathematical nonsense. He sounds quite impressive and brainy. But this isn't a course in film trivia, so back to the original question. Can anyone tell me why the Pythagorean Theorem is true?

Ernest: You mean give a proof?

Professor: Yes.

Ernest: We forgot all that proof stuff after the last test.

Sam: It's not like you need to know this to be president of the United States.

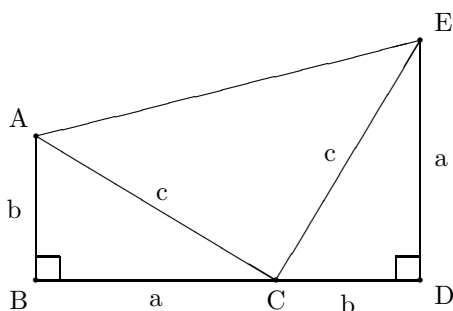
Professor: It's very interesting that you mention presidents. One of the first mathematical papers written by an American was by Garfield . . .

Ernest: (interrupting) The cat?

Professor: No, the President. And guess what he proved?

Beth: (tentatively) The Pythagorean Theorem?

Professor: You guessed it! His proof goes like this: [draws on the board]



Professor: Notice that the figure is composed of three triangles. Can anyone tell me what these three triangles are?

Paul: The original a - b - c right triangle appears on the left side of the diagram. It appears a second time on the right, but rotated 90 degrees.

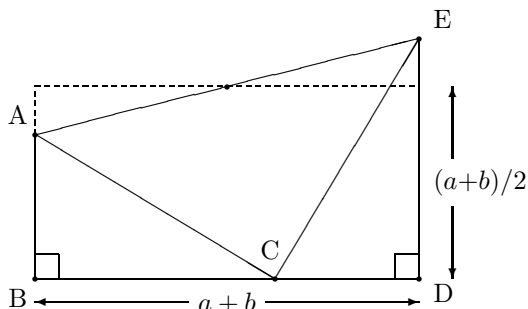
Professor: What about the middle triangle?

Sam: It looks like a square cut along the diagonal.

Professor: Very observant. Can anyone give a proof of Sam's observation that would make your high school geometry teacher proud?

Paul: I doubt it, but then I had a very fussy geometry teacher. Let's see. I need to label some things in the diagram. [The Professor hands her the chalk and she goes to the board.] Let's call the three triangles T_1 , T_2 , and T_3 , and let's call these three angles at the bottom $\angle 1$, $\angle 2$, $\angle 3$. Oh, here's $\angle 4$ in the left corner. [The diagram now looks like:]

Professor: That's right, it's a trapezoid. And we can do without the comments, Ernest. A trapezoid is a quadrilateral where exactly two sides are parallel. Notice that this trapezoid has the same area as a rectangle whose width is $a + b$ and whose height is the average of the height of the two parallel sides, $\frac{a+b}{2}$.



Professor: The equation for the area of this trapezoid is [writes on board]

$$\text{Trapezoid Area} = (a + b) \frac{a + b}{2}.$$

Since the area of the trapezoid is precisely the same as the total area of the three triangles, we get the equation:

$$(a + b) \frac{a + b}{2} = ab + \frac{1}{2}c^2.$$

Multiply this equation by 2 to clear denominators:

$$(a + b)(a + b) = 2ab + c^2.$$

Multiply $(a + b)(a + b)$ by FOIL:

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Subtract $2ab$:

$$a^2 + b^2 = c^2.$$

And voila, we have the Pythagorean Theorem. Not bad for a politician.

Ernest: Maybe he had more time on his hands than presidents do today. Like he didn't have to worry about Middle East crises or anything.

Professor: I don't really know what major problems faced President Garfield. I'm sure he was busy.

5.3 Exercises

1. If $AB = 7$ and $BC = 9$, what is the product of the lengths of the diagonals of the rectangle $ABCD$?

2. What is the length of a diagonal of a square whose perimeter is 100?
3. What is the area of an equilateral triangle of side measure a ?
4. A lattice point in the x - y plane is a point (m, n) whose x and y coordinates are both integers. Is it possible to find three lattice points A , B and C so that $\triangle ABC$ is an equilateral triangle. Hint: Use Pick's Theorem.

4. Similarity

A matching of the vertices of a first polygon with the vertices of a second polygon so that (i) corresponding angles are congruent and (ii) corresponding sides are proportional is called a **similarity**. The constant of proportionality must be the same for each side. We also say that the first polygon is **similar** to the second when there is a similarity between them. Roughly speaking, similar polygons have the same shape, but not necessarily the same size. If we enlarge a photograph to twice its original size we get a similarity.

Applying this definition to triangles, we say that two triangles $\triangle ABC$ and $\triangle XYZ$ are similar, written $\triangle ABC \sim \triangle XYZ$ if and only if

$$m\angle A = m\angle X \quad m\angle B = m\angle Y \quad m\angle C = m\angle Z$$

and

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}.$$

Obviously we do not need to do all this work to show that two triangles are similar. In fact it is enough just to determine that two corresponding angles are congruent.

Theorem 5.2 (Angle—Angle Similar Triangle Theorem). $\triangle ABC \sim \triangle XYZ$ if and only if

$$\boxed{m\angle A = m\angle X \text{ and } m\angle B = m\angle Y}$$

Notice that when two corresponding angles are congruent, it automatically follows that the third pair of corresponding angles are also congruent. [Why?] We will not prove this theorem, although the reader might wish to think about how it could be proved.

Another “obvious” theorem tells us that if length and width increase by a factor of r , then the area increases by a factor of r^2 . When we double length and width, then we quadruple the area.

Theorem 5.3 (Area of Similar Triangles). Suppose $\triangle ABC \sim \triangle XYZ$. If $r = \frac{XY}{AB}$, then

$$\boxed{\text{the area of } \triangle XYZ \text{ is } r^2 \text{ times the area of } \triangle ABC.}$$

A simple proof of this theorem involves simply noting that the altitudes of the two similar triangles are in the same constant proportion as the sides.

This theorem generalizes from triangles to polygons.

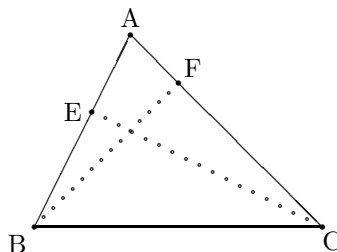
Theorem 5.4 (Area of Similar Polygons). Suppose Polygon 1 is similar to Polygon 2 and the ratio of proportionality is r . Then the ratios of their areas is r^2 .

We now present a proof of a theorem which we assumed to be true in order to define the area of a triangle.

Theorem 5.5. Given $\triangle ABC$. Let \overline{CE} be the altitude from vertex C to line \overleftrightarrow{AB} and let \overline{BF} be the altitude from vertex B to line \overleftrightarrow{AC} . Then

$$CE \cdot AB = BF \cdot AC.$$

Assume for now that E and F lie on the sides of the triangle, as shown in the following diagram.



It is easy to prove that $\triangle BFA$ is similar to $\triangle CEA$. Simply observe that

$$\angle BAF = \angle CAE = \angle A$$

and

$$m\angle BFA = m\angle CEA = 90.$$

By the Angle—Angle Similarity theorem, the two triangles are similar. Consequently,

$$\frac{CE}{AC} = \frac{BF}{AB}$$

which implies

$$\frac{1}{2}CE \times AB = \frac{1}{2}BF \times AC$$

and hence we get the same area for a triangle no matter which side we choose for the base.

5.4 Exercises

1. Show that the argument at the end of the section works even if E and F do not lie on the sides of the triangle.

2. Can you extend the similar Triangle Theorem to quadrilaterals? That is, if you match the vertices of $ABCD$ with $WXYZ$ so that corresponding angles are congruent, does it follow that the quadrilaterals are congruent?

3. Suppose that $\triangle ABC \sim \triangle DEF$.

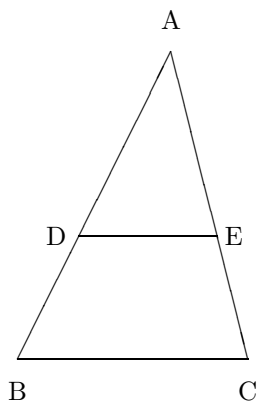
(a) If $AB = 2$, $BC = 7$, $CA = 6$, and $EF = 14$, what are FD and DE ?

(b) If $AB = 4$, $BC = 10$, and $DE = 7$, what is EF ?

(c) If $m\angle c = 90$, $AC = 5$, $AB = 13$, and $EF = 36$, what is DE ?

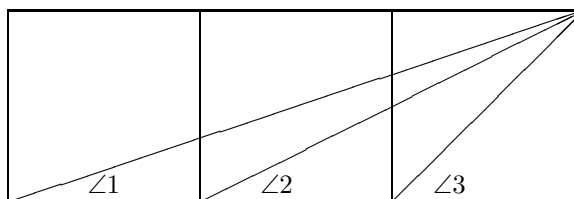
4. [The Divided Triangle]

Given a general triangle $\triangle ABC$. Suppose DE is a segment parallel to side BC such that the area of $\triangle ACD$ is exactly half of the area of $\triangle ABC$. What is the ratio of the length of segment AD to the length DB ?

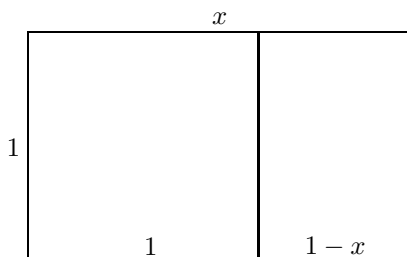


5. [The Three Angle Problem]

Given the three adjacent squares below, prove that $\angle 1 + \angle 2 = \angle 3$.



6. [The Golden Ratio] A 1×1 square is cut off of a $1 \times x$ rectangle:



In the above diagram, find the value of x that makes the rectangle on the right similar to the original rectangle. The number x is called the *golden ratio*. Discovered by the ancient Greeks, it appears in art, architecture, and in nature. For more information, do a google search on “golden ratio.”

7. [Fibonacci Numbers] Consider the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

These are called the **Fibonacci numbers** F_i .

- Give the rule for generating the n th Fibonacci number F_n from the previous two Fibonacci numbers.
- Draw these as areas of connected squares.
- Is F_n ever a square (bigger than 1)?
- Calculate the ratios F_n/F_{n-1} .
- Compare these ratios with the golden ratio you found in Exercise 6.