

Regular Polygons and Circles

May the circle be unbroken – Gabriel/Habershon

To go round and round and round in the circle game – Joni Mitchell

1. Regular Polygons

Definition 6.1. A **regular polygon** is a polygon whose sides are all the same length and whose interior angles all have the same measure.

An equilateral triangle is a 3-sided regular polygon; a square is a 4-sided regular polygon.

Theorem 6.1. The measure of each interior angle of a regular polygon with n sides is

$$\frac{(n-2)180}{n}.$$

Observe that for $n = 3$ and 4 this formula gives the correct angle measurements for an equilateral triangle and a square, viz. 60 and 90.

Definition 6.2. A **circle of radius r** with **center** at the point O is the set of all points P whose distance from O is r .

Our goal is to prove the following two theorems

Theorem 6.2. [circumscribed circle] A circle can be **circumscribed** about any regular polygon.

That is, there exists a circle C passing through each vertex of the regular polygon, so that the sides of the polygon all lie inside the disk with boundary C .

Theorem 6.3. [inscribed circle] A circle can be **inscribed** inside any regular polygon.

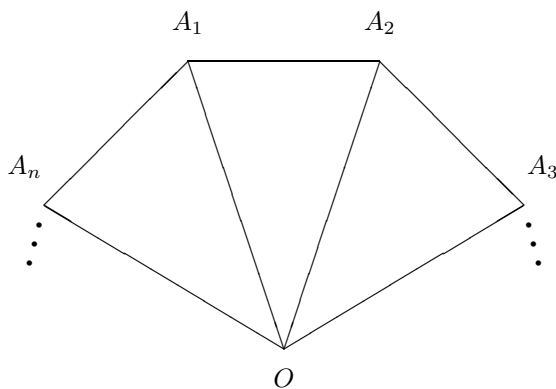
That is, there exists a circle C touching each side of the regular polygon, so that the circle lies inside the closed region whose boundary is the polygon.

Let's start with the Circumscribed Circle Theorem. Assume we are given an n -sided regular polygon $A_1A_2 \cdots A_n$. It is convenient to take A_{n+1} to be A_1 , so that $\overline{A_nA_{n+1}} = \overline{A_nA_1}$. Our first question is: where is the center of the circumscribed circle? It turns out that the angle bisectors of two consecutive angles of the polygon can be used to locate the center of the circumscribed circle.

Definition 6.3. The *center* of a regular polygon $A_1A_2 \cdots A_n$ is the intersection point O of the angle bisector of $\angle A_nA_1A_2$ and the angle bisector of $\angle A_1A_2A_3$.

Definition 6.4. A **radius** of a regular polygon is any line segment joining the center O to one of the vertices.

Definition 6.5. A **central angle** of a regular polygon is an angle formed by two consecutive radii of the polygon.



Proposition 6.4. $\triangle A_1OA_2$ is isosceles.

Problem 6.5. Show that

$$\triangle A_1OA_2, \triangle A_2OA_3, \dots, \triangle A_nOA_1$$

is a set of n congruent, isosceles triangles. (Be careful. You cannot just assume that $\overrightarrow{A_3O}$ is the angle bisector of $\angle A_2A_3A_4$.)

Conclude that

Proposition 6.6. Any two radii of a regular polygon are congruent.

and

Proposition 6.7. All central angles of a regular polygon are congruent.

Thus the circle C centered at O of radius $r = OA_1$ passes through every vertex of the polygon. Let D be the closed circular disk whose boundary is the circle C . Since D is convex, each line segment $\overline{A_i A_{i+1}}$ lies inside D . So C is the desired circumscribed circle about the regular polygon $A_1 A_2 \cdots A_n$. Our argument establishes the truth of Theorem 6.2. Moreover,

Theorem 6.8. The measure of the central angle of a regular polygon with n sides is

$$\frac{360}{n}.$$

We now consider the Inscribed Circle Theorem. It turns out that the centers of the inscribed and circumscribed circles of a regular polygon are the same. So we can use O as the center of the inscribed circle.

Definition 6.6. An **apothem** of a regular polygon is any line segment joining the center O and the midpoint of one of the sides.

Let M_i denote the midpoint of the segment $\overline{A_i A_{i+1}}$.

Proposition 6.9. The apothem $\overline{OM_1}$ is perpendicular to the side $\overline{A_1 A_2}$.

Problem 6.10. Show that

$$\triangle A_1 O M_1, \triangle A_2 O M_1, \triangle A_2 O M_2, \triangle A_3 O M_2, \dots, \triangle A_n O M_n, \triangle A_1 O M_n$$

is a set of $2n$ congruent, right triangles.

Conclude that

Proposition 6.11. All apothems of a regular polygon are congruent.

The circle C' centered at O of radius $r = OM_1$ is the desired inscribed circle, thereby completing the proof of Theorem 6.2.

We conclude with a theorem relating the area inside a regular polygon to its perimeter.

Theorem 6.12. [area theorem] The area A of a regular polygon is given by the formula

$$A = \frac{1}{2} \ell P,$$

where ℓ is the length of an apothem and P is the perimeter of the polygon.

6.1 Exercises

1. Prove Proposition 6.4
2. Do Problem 6.5.
3. Prove Proposition 6.9
4. Do Problem 6.10.
5. Explain why the circle C' lies *inside* the closed region whose boundary is the regular polygon $A_1A_2 \cdots A_n$.
6. Show that the perpendicular bisectors of two consecutive sides of the polygon can also be used to locate the center of the inscribed circle.
7. Prove the area theorem (Theorem 6.12).
8. Prove that given any three noncollinear points, there is a circle C which passes through all three points.

It is well known that Renaissance painters and architects frequently used the proportion of the golden ratio (see Exercises 5.5, problem 6) in their art. A colleague of the author once attended a thesis defense where an art student presented his discovery that Renaissance paintings have another, less known, geometric property. The student found that when you select the proper three points, such as center of the face, abdomen, and hand, of a painting from the Renaissance period, it turns out that all three points lie on a common circle. Critique this “discovery” in light of the geometry result you just proved in this exercise.

2. Circles

Every school child knows the formulas for circumference and area of a circle. The most common math joke, no doubt, is the following conversation between a school boy and his father:

Father: What newfangled ideas are you learnin' in your geometry class?

Boy: Teacher taught us today that pi r squared.

Father: What's wrong with your teacher? Everybody knows pies are round, not square.

Few students enter college, however, knowing *why* these formulas are true or *how* they are related.

If we draw a circle inside a square, it is clear that the circumference of the circle is proportional to the perimeter of the square and the area of the circle is proportional to the area of the square. The question is:

Why are the two constants of proportionality the same?

We can use π for the ratio of circumference to diameter

Circumference formula: $C = 2\pi r.$

and k for the ratio of area to the square of the radius

Area formula: $A = kr^2.$

We will show that $k = \pi$.

Argument 1. [unrolling an annulus.] Recall the formula for the area of a trapezoid:

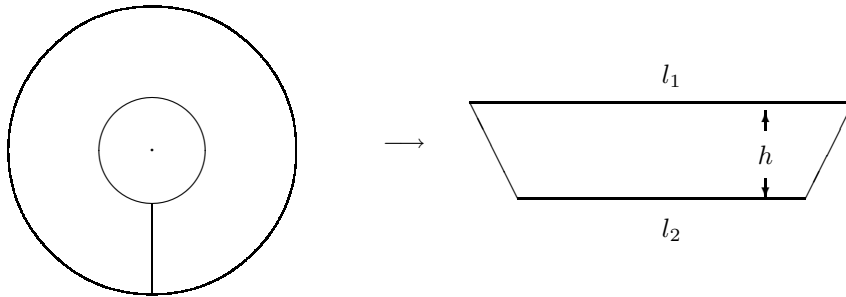
Theorem 6.13. Let ABCD be a trapezoid where \overline{AB} and \overline{CD} are parallel. Let h be the distance from A to \overline{CD} as measured along a ray perpendicular to \overline{CD} . Then the area A of the trapezoid is given by

$$A = h \frac{AB + CD}{2}.$$

Consider two concentric circles, the smaller one of radius r and the larger one of radius $\sqrt{2}r$. By (2) the area of the big circle is $k(\sqrt{2}r)^2 = 2kr^2$, while the area of the small circle is kr^2 . Subtracting these areas gives the area of the annulus between the two circles

$$A_{ann} = kr^2.$$

Unroll the annulus to get a trapezoid:



The length l_1 of the top side of this trapezoid is the circumference of the big circle and the length l_2 of the bottom side is the circumference of the little circle. By (1)

$$l_1 = 2\pi\sqrt{2}r \quad \text{and} \quad l_2 = 2\pi r.$$

The height of the trapezoid is $h = \sqrt{2}r - r = (\sqrt{2} - 1)r$. Using the formula for the area of a trapezoid, we find

$$A_{trap} = h \frac{l_1 + l_2}{2} = (\sqrt{2} - 1)r \frac{2\pi\sqrt{2}r + 2\pi r}{2} = (\sqrt{2} - 1)(\sqrt{2} + 1)\pi r^2 = \pi r^2.$$

Since $A_{ann} = A_{trap}$, we conclude that $k = \pi$.

Argument 2. [Proof by exhaustion.] Consider the formula for the area of a regular polygon inscribed inside a circle of radius r :

$$A = \frac{1}{2}\ell P,$$

where ℓ is the length of an apothem and P is the perimeter of the polygon. As the number of sides n gets larger and larger, it is easy to see that

$$\ell \longrightarrow r$$

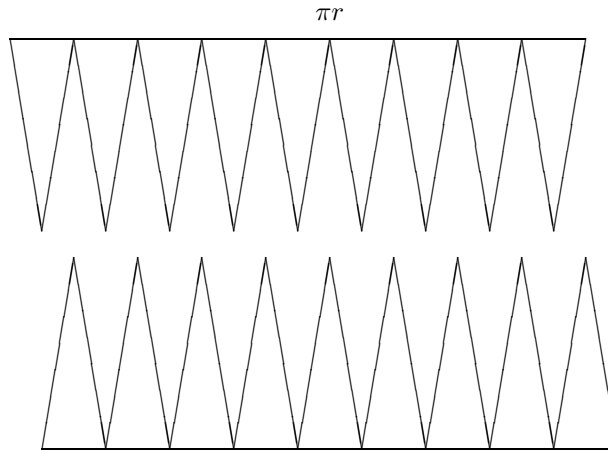
and

$$P \longrightarrow C,$$

the circumference of the circle. Since $C = 2\pi r$, the area of the regular inscribed n -gon gets closer and closer to

$$\frac{1}{2}r \cdot 2\pi r = \pi r^2.$$

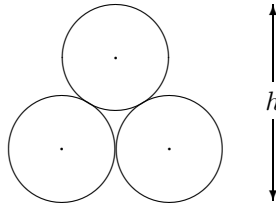
Argument 3. [The teeth argument.] Divide the disk of radius r into two semidisk. Cut each semidisk along n equally spaced radii. Fan out the sections to form two “sets of teeth”:



When the upper and lower “teeth” are meshed together, they form a “rectangle” of dimensions r by πr . Thus the area of the circle has been transformed to a rectangle of area πr^2

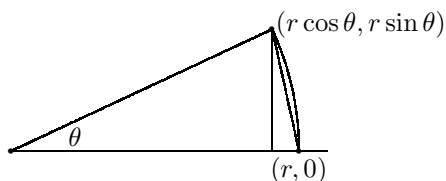
6.2 Exercises

1. What is a better deal: a 10 inch pizza for \$7 or a 14 inch pizza for \$10?
2. Three pipes, each of radius 4 inches, are stacked as shown.



What is the distance from the ground to the top of the stack?

3. A windshield wiper rotates through a 120 angle as it cleans the windshield. The rubber part of the wiper blade is 14 inches long and connects to the wiper arm at a point 4 inches from the pivot point (and this extends to a point 18 inches from the pivot point). What is the area cleaned by the windshield wiper?
4. A goat is tied to a 60 by 40 foot barn. The rope is attached to the barn at a point exactly 6 feet east of the southwest corner of the barn. The length of the rope is 12 feet. Over what area can the goat graze?
5. Show how to transform the circle into an isosceles triangle whose altitude has length r and whose long side has length $2\pi r$. What is the area of this triangle? Does this give another proof that the area of a circle is πr^2 ?
6. Discuss the rigor of Arguments 1 – 3.
7. [A Calculus Argument] We present a fourth argument that $A = \pi r^2$ using calculus. The area of a regular polygon inscribed inside a circle of radius r can be determined by using a little trig. Let us assume that the center of the circle is the origin $(0, 0)$. The central angle of each section is $\theta = \frac{2\pi}{n}$ radians. We can rotate the polygon so that the first section has vertices $(0, 0)$, $(r, 0)$, and $(r \cos \theta, r \sin \theta)$.

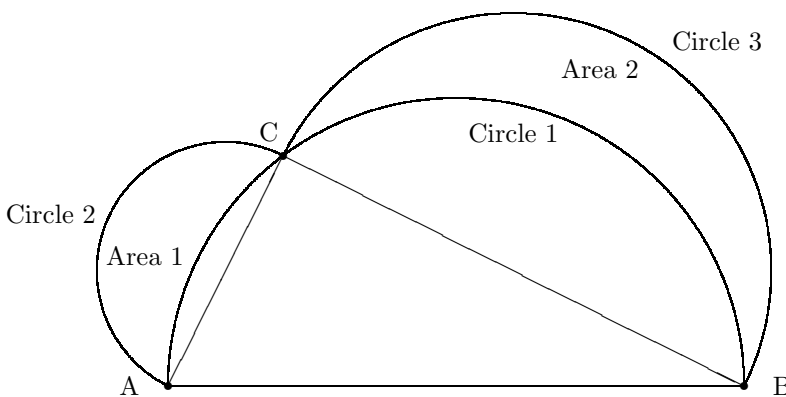


The area of a triangle with height $h = r \sin \theta$ and base $b = r$ is $\frac{1}{2}r^2 \sin \theta$. Since the regular n -sided polygon consists of n congruent triangles glued on their edges, its area is n times the area of each triangle:

$$A = \frac{1}{2}nr^2 \sin \theta = \pi r^2 \frac{\sin \theta}{2\pi/n} = \pi r^2 \frac{\sin \theta}{\theta}.$$

- (a) Evaluate this formula for $n = 8$ ($\theta = \frac{\pi}{4}$ radians) and for $n = 12$ ($\theta = \frac{\pi}{6}$ radians).
 (b) When $r = 1$, your answer in (a) gives two approximations for π . What are they?
 (c) As n approaches infinity, the angle θ approaches 0, and hence the area of the regular polygon approaches $\pi r^2 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$. Using the limit equation $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ from calculus gives us a fourth derivation of the formula for the area of a circle. There is one major flaw with this “proof.” If you examine the derivation of the limit formula for $\frac{\sin \theta}{\theta}$ in any calculus book, you will discover that their proof assumes that we already know the area of a circle is πr^2 (in order to compute the area of the circular sector). In mathematics, we call such an argument circular reasoning (no pun intended).

8. [Lunes] Given the following diagram:



Hypotheses:

\overline{AB} is a diameter of Circle 1

\overline{AC} is a diameter of Circle 2

\overline{BC} is a diameter of Circle 3

Area 1 = the area between Circle 2 and Circle 1

Area 2 = the area between Circle 3 and Circle 1

Prove: Area 1 + Area 2 = Area of $\triangle ABC$.

3. Tangents and Inscribed Angles

Definition 6.7. A **tangent** to a circle is a line that intersects the circle at exactly one point, called the point of tangency.

Consequently all points of the tangent line, other than the point of tangency, lie in the exterior of the circle. This observation makes it straight-forward to prove the following theorem, which asserts that a tangent line is perpendicular to the radius at the point of tangency.

Theorem 6.14. Let ℓ be a line tangent to a circle with center O and let P be the point of tangency of ℓ and the circle. Then the radius \overline{OP} is perpendicular to ℓ .

Definition 6.8. An angle $\angle ABC$ is said to be **inscribed** in an arc of a circle if its sides contain the endpoints of the arc and its vertex is a point of the arc other than an endpoint.

Theorem 6.15. Let A , B , and C be three distinct points of a circle whose center is O . Assume that B lies on the major arc joining A and C . Then

$$m\angle ABC = \frac{1}{2} m\angle AOC.$$

To prove this theorem, consider 3 cases, based on where the center of the circle lies in relation to the triangle. There are three distinct possibilities:

Case 1. The center of the circle lies on side \overline{BC} . (The case where the center lies on \overline{BA} is similar.)

Case 2. The center of the circle lies in the interior of $\angle ABC$.

Case 3. The center of the circle lies in the exterior of $\angle ABC$.

6.3 Exercises

1. Fill in the details of the proof of Theorem 6.14. Here is an outline of the proof. Suppose, towards a contradiction, that OP is not perpendicular to the tangent line ℓ . Then construct

a line m through O which is perpendicular to ℓ . Line m meets ℓ at a point F , the foot of the perpendicular. Now argue that $OP > OF$ by considering the right triangle $\triangle OPF$. What does this inequality say about the location of F : is F inside or outside the circle? You should see an immediate contradiction.

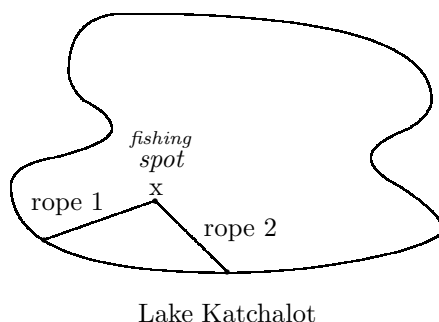
2. If you have had calculus, give a proof that the tangent line to the circle $x^2 + y^2 = a^2$ is perpendicular to the radius by using derivatives and slopes.
3. Fill in the details of the proof of Theorem 6.15, considering each of the three cases.
4. Use Theorem 6.15 to prove the following

Theorem 6.16. Assume that the four corner points of the quadrilateral $ABCD$ lie on a circle. Suppose that the two diagonals \overline{AC} and \overline{BD} are perpendicular and meet at the point P inside the circle. Let ℓ be the line through P and perpendicular to \overline{BC} . Prove that ℓ bisects \overline{AD} .

5. The astute reader will have noticed that we did not define the term **arc** of a circle. A definition of the arc connecting two points A and B on a circle obviously means all the points *between* A and B . Once again we use that word “between,” although this time our betweenness is between points on a circle, not a line. We run into a second problem in that there are two circular paths which connect A and B . We call the shorter path the *minor arc*. Draw some pictures and try to formulate a reasonable definition of (minor) arc between A and B . How can we distinguish between the minor and major (longer) arcs?

4. Circles and Navigation

Professor Flappenjaw has brought three pieces of rope to his class. One piece measures 30 feet long and the other two are 10 feet in length. He moves the chairs away from the center of the classroom and lays the 30 foot rope in a curved shape on the floor, like this:



Professor: Alright, class, I want you to think of this rope as the edge of a lake, where you go fishing in the summer.

Ernest: You're making me wish that it was summer already.

Professor: Just imagine, Ernest, that you discover a really great fishing spot in the lake and you want to mark it in some way so that you can come back to it tomorrow. How would you do that?

Ernest: Well, you certainly can't put a big X on the water. Though maybe you could mark an X on a board and float the board over the spot.

Sam: The winds and waves would just blow your board to a different location.

Professor: I'm afraid Sam has a point, Ernest. You need something besides a board to work with. Suppose I gave you a long rope, with one end attached to the shore. Can any one think of a way to use the rope?

Hedda: I got it! You could tow the rope behind the boat till you get to your fishing spot, then pull the rope tight and make a mark on the rope right above the spot. The next day all you gotta do is row until the rope is extended from the shore to the mark you made the day before.

Paul: But how do you know you rowed the boat in the right direction?

Professor: Alright, let's try it. Suppose my favorite fishing spot is here. [He drops a quarter on the floor.] This is how golfers mark their balls.

Class snickers.

Professor; [slightly flustered] Obviously, I meant this is how they mark their *golf* balls when they're putting on the greens. Now settle down. I need a volunteer to hold one end of the rope attached on the shore.

Sam: I can do it, Professor. It's good to be useful.

Professor: Thanks, Sam. Now we mark the rope right here. [He pulls the rope tight over the quarter on the floor and marks the rope with a felt tip pen.] Now I'm going to hide the fishing spot with my sweater. [He drops his sweater over the quarter.] What happens when you try to use the marked rope to find the exact same fishing spot tomorrow?

Ernest: Paul's right. It's not enough to know how far to go. You also need to know the right direction.

Professor: What is the geometric shape formed when I hold the rope taut and vary the direction? [The professor swings the rope, while Sam holds the other end fixed.]

Paul: It's obviously an arc of a circle.

Professor: So how can I decide which point on this arc is the spot where I was fishing yesterday?

Steve: Maybe another rope would help.

Professor: Very good. Suppose we had a second rope tied at another point on the shore. Do I see another volunteer?

Hedda: [Picks up the other 10 foot rope.] I'm way ahead of you.

Professor: [Removes his sweater from the floor, revealing the fishing spot.] Now, Ernest, mark this second rope at the spot we want to remember. [Ernest makes the mark.]

Professor: If I uncoil both ropes and then come back tomorrow, what happens when both ropes are extended to their marks?

Steve: They meet right at the spot.

Professor: Can anyone explain our success mathematically?

Paul: When each rope is taut, you get two arcs, which intersect at a unique point.

Professor: Are you sure?

Paul: I guess two circles intersect in *two* points, but the other point of intersection would be on the land, unless the lake was shaped like a horseshoe.

Professor: Who knows what GPS stands for?

Steve: Global Positioning System.

Ernest: I thought it meant Global Position Satellite.

Professor: Steve's right, Ernest, although the technology does rely on satellite information. And what does GPS do?

Ernest: It tells your location somehow.

Professor: How does it do that?

Sam: I think I see what you're driving at. GPS measures the distance from the unit to different satellites in the sky. Your position, the place to be located, is like our fishing spot. The satellites are like the ends of the two ropes and the distances to the satellites are like the lengths of the ropes.

Professor: Exactly. Except, of course, with GPS the location is in 3-dimensional space, not just a spot on a 2-dimensional lake. In space you need at least three "ropes" to locate an object.

Ernest: Wait till I tell my dad that I know how satellite navigation systems work.

6.4 Exercises

1. Pinpoint location assumes that all distance measurements are perfect, when in fact, they could differ by a small amount from the true value. Suppose the two ropes in the Professor's example are anchored 5 feet apart. If the length of rope 1, from the fixed end to the mark, is between 3 and 3.01 feet long and the length of rope 2 is between 4 and 4.01 feet to the mark, describe geometrically the smallest region which you can be certain contains the fishing spot?
2. In 3-dimensional space, the set of all possible points whose distance from a fixed point is a constant value r forms a sphere of radius r . Describe geometrically the intersection of two spheres. What is the intersection of three spheres?

5. Platonic Solids

Definition. A platonic solid is a convex solid whose faces are all congruent regular polygons. The faces must be joined in such a way that the polyhedral angles—the angle formed where three (or more) faces meet at a common vertex—are congruent.

For example, a cube is a platonic solid. The faces are squares and the polyhedral angles all have measure 90.

Our goal is to determine all possible platonic solids. We begin with some variables:

V = number of vertices

E = number of edges

F = number of faces

s = number of sides on each face

n = number of edges that meet at each corner

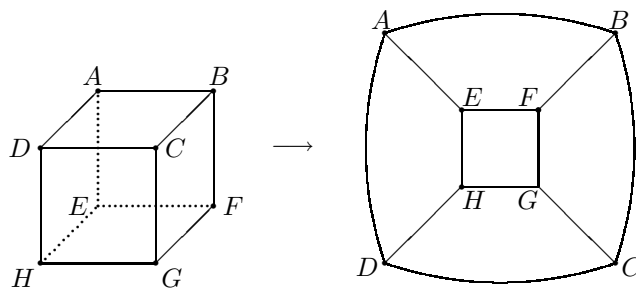
Now complete the following ten-step program:

Step 1. Explain why: $s \geq 3$ and $n \geq 3$

Step 2. Explain why: $sF = 2E$

Step 3. Explain why: $nV = 2E$

Think of the skeleton of a platonic solid, that is, its vertices and edges, as made of rubber band material. Now suppose we place the solid so that one of the faces is on the top. Then we can remove this top face and stretch the rubbery graph and then flatten it onto a plane. Thus, the skeleton of every platonic solid is topologically equivalent to a connected polar graph. For example, flattening the cube looks like this:



Since the resulting graph (of vertices and edges) is a connected planar graph, Euler's formula holds. In this case, the number of regions is precisely the number of faces. The removed top face becomes the outside region.

Step 4. Euler's Formula: $F + V - E = 2$

Step 5. Using Steps 2, 3, and 4, derive:

$$\frac{2E}{s} + \frac{2E}{n} - E = 2$$

Step 6. Rearrange the above equation to obtain:

$$\frac{1}{E} = \frac{1}{s} + \frac{1}{n} - \frac{1}{2}$$

Step 7. Explain why Step 5 implies

$$\frac{1}{s} + \frac{1}{n} > \frac{1}{2}$$

Step 8. Show that

- (i) if $s = 3$, then the only choices that satisfy the inequality in Step 7 are $n = 3, 4$, or 5 ;
- (ii) $s = 4$, then the only choice that satisfy the inequality in Step 7 is $n = 3$;
- (iii) $s = 5$, then the only choice that satisfy the inequality in Step 7 is $n = 3$.

Step 9. Once you know s and n , you can use the equation in Step 6 to compute E . Then the equations in Facts 2 and 3 easily give the values of F and V :

$$F = \frac{2E}{s} \quad \text{and} \quad V = \frac{2E}{n}$$

For example, when $s = 4$ and $n = 3$, Step 6 gives us

$$\frac{1}{E} = \frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{1}{12},$$

so $E = 12$, and therefore, $F = 24/4 = 6$ and $V = 24/3 = 8$. The platonic solid with 6 faces, 8 vertices, and 12 edges is, of course, the cube.

Step 10. Complete the chart to obtain a list of

The Five Platonic Solids

Name	s	n	F	V	E
tetrahedron	3	3			
octahedron	3	4			
icosahedron	3	5			
cube	4	3	6	8	12
dodecahedron	5	3			

The names listed in the table are based on the number of faces. If you are wondering, a cube is also called a hexahedron, since it has six faces.