MODULE 0

Overview

This course is designed as an introduction to geometry, especially for elementary education majors who plan to teach middle school mathematics.

Calculatus Eliminatus

Before detailing the goals of the course, let me tell you what this course is not, following the search technique "calculatus eliminatus" coined by Dr. Seuss in the *Cat in the Hat*, which means *to discover what something is, you must discover what it's not.* This course is not a survey of geometry, nor is it a fast paced high school geometry course. We do not concentrate on analytic geometry—so useful for calculus, nor do we examine the many beautiful and spectacular theorems not typically presented in a standard high school plane geometry course. It is not a foundations course, where we lay down a correct and consistent set of axioms for neutral, Euclidean, or non-Euclidean geometry. It is not a class designed to teach math majors how to make correct proofs. Finally, it is not a methods course in how to teach geometry.

"Why" versus "how to"

This course is more of a "why course" than a "how to" course. A student who has been through the public educational system in the United States no doubt has been exposed to the "how to" method of mathematics education. In grade school you learned how to add, subtract, multiply, and divide. You learned how to reduce fractions and how to find percentages. In algebra you learned how to solve equations, factor polynomials, and set up word problems. In geometry you learned how to measure angles and construct perpendiculars. All of these are important and essential techniques. Our approach is to revisit many of the well remembered facts from geometry and ask why are they true. The reader may try her or his hand on the following "why" questions:

- Why can't you divide by zero?

• Why does the standard method for multiplying two 2-digit numbers involve addition? After all, we're multiplying, not adding.

• Why is the square root of x written as $x^{\frac{1}{2}}$?

• Why is a negative times a negative equal to a positive?

• Why does a line segment have length since it is composed of points of zero width?

It is our intention to question many things we have taken for granted for years. If, for example, two angles of a triangle have degree measurement 45 and 70, then you find the measure of the third angle by subtacting the sum of 45 and 70 from 180 to get the answer 65 degrees. This *how to* procedure taught in grade school is based on a well known mathematical theorem: the sum of the three angles of a triangle add to 180 degrees. Alright, reader, what makes this theorem true? It may interest you to learn that this theorem is not true for triangles drawn on a sphere. Draw a triangle on the surface of the earth as follows: start at the North Pole, follow a longitude to the equator, travel along the equator for one quarter of the circumference of the earth, and then return to the North Pole along another longitude. Guess what? The three angles of this triangle all have measure 90. "Wait a minute," you say, "this means that the sum of these three angles is 270." That's right. So why is the 180 degree theorem true on a flat plane but not on a sphere?

Precise Definitions

We are particularly interested in giving precise definitions for simple mathematical objects and ideas. The truth is that formulating precise definitions is not always easy. For example, I would be hard pressed to give a definition of "tree" that would satisfy a biologist, and yet I know a tree when I see one. Similarly, I think you would find it difficult to define the number 2, although I'm sure you know what it is. Or at least you know what its properties are and how to do arithmetic with it. Some examples from geometry:

• What are points and lines?

• What is distance?

• Can two points on a line touch each other?

• Given three points on a line, how do we know which one is in the middle?

• What are angles?

• What is area?

Another goal of the course is to state and prove the really famous theorems of geometry whose truth we have come to accept as a matter of faith, such as the 180 degree theorem mentioned above or the well loved Pythagorean Theorem.

• Why does the area of a triangle equal one half base times height? Why don't you get three different answers, depending on which side you use for the base?

• Why does the area of a circle equal pi times radius squared? What is π anyway?

Why, why, why are all these true?

Try to remember what it was like to be a five year old, asking an endless stream of "why" questions, before grownups "cured" you of the habit with the all-encompasing answer: "just because," or even worse, "because I say so."

Some sideroads

A main goal of this course is to increase the students' awareness and appreciation of the many diverse areas of mathematics related to geometry which are not usually encountered in high school or even undergraduate math courses. Besides the standard topics of elementary geometry—such as triangles, polygons, circles, congruence, similarity, area—we discuss topics in analysis, logic, and topology. Since lines are such an important part of geometry, we spend a lot of time trying to understand the prototype of all lines, namely the real number line.

The course begins with the question "Does $.999\cdots$ equal 1?" We then launch into a discussion of infinity. The concept of infinity intrigues many people. It is mysterious. Car companies are named after it. You never hear of a car named "Square Root" or "Logarithm." Can we compare infinite sets? If you got a dollar for every positive integer and I got a dollar for every point on the real number line, who would be richer? Or are infinity-aires all equally wealthy?

Logic is typically taught either as a prelude or else as an appendix to a course in geometry. I think it is important enough to have its own section. There are precise meanings to such innocent little words as *and*, *or*, and *implies.* For example, does an *or* question allow both alternatives to be true? The answer is "no" if you are offered coffee or tea, but "yes" if you are offered cream or sugar. What about the statement "Give me liberty or give me death!"?

The third chapter of the book examines topology, sometimes called "rubber-sheet geometry." We are allowed to bend, stretch, shrink, and twist topological objects, but we are not allowed to tear, cut, or glue them. The chapter starts out with a precise definition of distance, which we use to define other very basic concepts such as "inside," "outside," "boundary," and "connectedness." Many of these ideas enable you to use the logic you just learned in the previous section and to play mathematician in a simple and safe environment.

The Main Highway: Geometry

After our excursions into analysis, logic, and topology, we return to the main highway: geometry. We define what points and lines are by specifying how they behave. As an analogy, a knight's moves on the chessboard are restricted by certain rules. In geometry such rules are called axioms. One axiom, for example, specifies that a unique line passes through any pair of distinct points. At the root of our system of geometry lie some very basic concepts. For example, what is a triangle? Answer: the three segments joining three non-collinear points. What is a segment? Answer: the points lying between two endpoints. Aha, we have arrived at the basic underlying concept: betweenness. What does it mean to say that one point is between two others? Similarly, what is an angle? How do we measure angles? How do we measure distance? All these concepts are fundamental to laying down a solid foundation for geometry. Just as all homeowners want their basement walls to not leak and to hold up the house, we want our geometry foundation to be logically sound and strong enough to support such advanced topics as perpendiculars, parallels, polygons, area, and similarity. We spend some time with the Pythagorean Theorem and we extend the number of right triangles we know with integer sides from two examples (3–4–5 and 5–12–13) to infinitely many. Regular polygons and circles are important enough to deserve an entire chapter to themselves. All the "pi stuff" comes in here.

One thread that run throughout our story of geometry involves several two thousand year-old questions that perplexed the ancient Greek mathematicians who first wrote down an axiomatic treatment of geometry. The first question is very simple: do parallel lines exist? If extended indefinitely, do the two rails of a railroad track ever come together? We can, like the Greeks, create an axiom which says they never meet, but mathematicians for centuries thought that such a statement should really follow as a theorem from the other, simpler axioms. With the discovery of non-Euclidean geometries we know now that the obvious and intuitive first impression is not always the truth. Geometries can exist which have no parallel lines or else lots more than expected. The Greeks would have never dreamed how weird it can get. Another problem that perplexed the Greeks and future geometers involves constructions with a compass (to draw circles) and a straightedge (to draw lines)—the so called "instruments of math construction." It is child's play to bisect an angle using a straightedge and compass. Try trisecting an angle using the same two tools. Bet you can't. It turns out that nobody can trisect an angle with just a compass and a straightedge. If you stick with the course you'll find out why.

General goals

Some general goals of the course are to gain an understanding and appreciation of

- reasoning in general and mathematical reasoning in particular
- the logical relationships among various geometric concepts
- how to construct, analyze, and place in context specific examples and facts
- how to infer general conjectures from the study of examples
- how to articulate and communicate problem solutions, verbally and in writing
- the use of abstraction in mathematics
- connections between geometry and art.
- and to have some fun.

Attitude counts

A student once told me that he and his roommate spent an hour debating about the 999 question, the first topic of the course. Perhaps the ultimate goal of the course is to get you to accept mathematics as something worth discussing with your friends. Many of you will someday teach young minds the mysteries of mathematics, and I hope your goal is not to enlist more members into the ever popular "I hate Math" Society—even if you have at times been a member yourself. Many ideas of mathematics can be really cool, if you just let it happen.