A partition of a positive integer $n$ is a sum of positive integers taken from a certain set. We do not care about the order in which we write the sum. For the number $n = 6$, the partitions $1 + 2 + 3$, $2 + 3 + 1$, and $3 + 2 + 1$ are all the same.

Odd parts Here we are interested in partitions of a number where the summands are required to be odd positive integers. For example, $n = 9$, we are only allowed to use 1, 3, 5, 7, and 9. So the partitions into odd parts are:

- $9$
- $7 + 2$
- $5 + 3 + 1$
- $5 + 1 + 1 + 1 + 1$
- $3 + 3 + 3$
- $3 + 3 + 1 + 1 + 1$
- $3 + 1 + 1 + 1 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

The total number of ways of writing $n$ as a sum of odd integers is

$$p_{\text{odd}}(9) = 8.$$  

Distinct parts Here we are interested in partitions of a number where the summands are required to be distinct positive integers. For example, $n = 9$, we may use any integer 1 through 9 as along as we do not use it more than once. So the partitions into distinct parts
are:

\begin{align*}
9 \\
8 + 1 \\
7 + 2 \\
6 + 3 \\
6 + 2 + 1 \\
5 + 4 \\
5 + 3 + 1 \\
4 + 3 + 2
\end{align*}

The total number of ways of writing $n$ as a sum of distinct integers is

$$p_{\text{distinct}}(9) = 8.$$ 

Key idea #106. Every positive integer has a unique representation as

$$m = 2^k \cdot \text{odd}.$$ 

This is a weaker statement than the prime power decomposition theorem that states that every number can be factored into a unique (up to order) product of primes.

In our list of partitions with odd parts, use multipliers for the number of times each odd part is repeated:

\begin{align*}
9 \\
7 + 2 \\
5 + 3 + 1 \\
5 + 4 \cdot 1 \\
3 \cdot 3 \\
2 \cdot 3 + 3 \cdot 1 \\
3 + 6 \cdot 1 \\
9 \cdot 1
\end{align*}

Now write each of the multipliers in binary, as a sum of power of 2’s:
Multiply these out:

9
7 + 2
5 + 3 + 1
5 + 4 \cdot 1
(2 + 1) \cdot 3
2 \cdot 3 + (2 + 1) \cdot 1
3 + (4 + 2) \cdot 1
(8 + 1) \cdot 1

Note that every permutation using odd parts is transformed into a permutation whose parts are distinct.

Exercise #107. Write down all the partitions of \( n = 10 \) using (a) odd parts; (b) distinct parts. Use the Key Idea to find a correspondence between these two lists of partitions.

In mathematics, a guess about a possible theorem based on numerical evidence (or sometimes just intuition) is called a conjecture.

Exercise #108. State a conjecture about the number of partitions of \( n \) into even parts and the number of partitions of \( n \) into distinct parts.
There are other examples of interesting results like the even–distinct connection. For example, suppose we restrict the parts to numbers congruent to 1 or 4 mod 5; these are numbers which end in 1, 4, 6, or 9. The corresponding partitions of \( n = 9 \) are

\[
\begin{align*}
9 \\
7 + 1 + 1 \\
6 + 1 + 1 + 1 \\
4 + 4 + 1 \\
4 + 1 + 1 + 1 + 1 + 1 \\
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
\end{align*}
\]

Now suppose, instead, that we make the restriction that the parts are not only distinct, but that any two parts must differ by at least 2. The corresponding partitions of \( n = 9 \) are

\[
\begin{align*}
9 \\
8 + 1 \\
7 + 2 \\
6 + 3 \\
5 + 3 + 1
\end{align*}
\]

We are not allowed to use \( 6 + 2 + 1 \), for example, because even though the parts are distinct, 2 and 1 do not differ by at least 2.

**Exercise** #109. Write down all the partitions of \( n = 10 \) using (a) numbers which are congruent to 1 or 4 mod 5; (b) distinct parts which differ by at least 2.

**Exercise** #110. State a conjecture about the number of partitions of \( n \) into parts congruent to 1 and 4 mod 5 and the number of partitions of \( n \) into distinct parts which differ by at least 2.