Clock Arithmetic

It's 6 o'clock at night. You eat supper for one hour, watch TV for 2 hours, study for 4 hours. What time are you done studying? If we add the time intervals, we get $6+1+2+4=13$. But 13 o'clock is not a valid time (except in the army). The time will be 1 in the morning, which we get by subtracting 12 from 13. In mathematics we write this as:

$$13 \bmod 12 = 1.$$

On a standard 12-clock, once you reach the number 12, the time resets to 0. Similarly, $25 \bmod 12 = 1$ and $-2 \bmod 12 = 10$, that is, 2 hours *before* 12 is 10 o'clock.

We are familiar with different clocks besides the 12-clock. For example, we can think of the days of the week as lying on a 7-clock. It is convenient to assign the days the values: Sun = 0, Mon = 1, Tues = 2, $\cdots$, Sat = 6. Suppose you begin a trip for Florida on Wednesday. You spend 2 days driving. You stay in a hotel in Orlando for 6 days, and a hotel in Miami for 5 days. It takes 2 days to drive home. On what day do you return home? Since Wednesday is day number 3, we can add up all the days to get: $3+2+6+5+2=18$. On a 7-clock, 18 is the same as 4, which we write as:

$$18 \bmod 7 = 4.$$

So you return on day number 4, or Thursday.

The number of pennies needed in the exact change for a purchase can be computed using a 5-clock. For example, suppose you buy a \$1.99 sandwich, a \$.89 drink, and the tax is 26 cents. How many pennies do you need to pay the bill? Adding the three numbers as cents, we have

$$(199+89+26) \bmod 5 = 314 \bmod 5 = 4.$$

You need 4 pennies (plus a dime and 3 dollars).

In general, on an n-clock we write

$$a \bmod n = r$$

if r is the remainder when you divide a by n. We also say "a reduces to r mod n."

Here are the addition and multiplication tables for 5-clock arithmetic:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Let's examine the times table. The zero row and zero column consists of all 0's. What did we expect? Zero times anything is zero. If we ignore the 0 row and column, the rest of the times table has some interesting properties. Notice, for example, the numbers 1, 2, 3, and 4 are *scrambled* when we multiply by 2, 3, or 4. That is, each of the rows list the numbers 1, 2, 3, 4 in some order. In the second row we get 2, 4, 1, 3; in the third row we get 3, 1, 4, 2; in the last row the numbers are backwards 4, 3, 2, 1. This scrambling phenomenon is the key idea in constructing the secret codes discussed in the next section.

Exercises:

1. Your birthday is on Monday this year. On what day will your birthday occur next year? Note that the answer depends on whether there is a leap year (giving an additional calendar day February 29) in the upcoming year.

2. Compute

$$\begin{aligned}(a)&\ (11+23-5) \bmod 7\\ (b)&\ 1000 \bmod 7\\ (c)&\ -250 \bmod 33\end{aligned}$$

3. To compute $(100\times 50+47) \bmod 11$, we first compute $100\times 50+47 = 5047$ and then find the remainder of 5047 when divided by 11. Since $5047 = 11\times 458+9$, we find $(100\times 50+47) \bmod 11 = 9$. Now is it possible to compute the remainders of the numbers 100, 50, and 47 first, and then use these in the calculation? Since $9\times 11 = 99$, it is easy to see that $100 \bmod 11 = 1$. Similarly, $4\times 11 = 44$, so $50 \bmod 11 = 6$ and $47 \bmod 11 = 3$. If we compute $(100\times 50+47) \bmod 11$ by using the remainders for 100, 50, and 47, we get $(1\times 6+3) \bmod 11 = 9$, the correct answer. What rules does this suggest about clock arithmetic?

4. Construct the + and × tables for the (a) 7-clock; (b) 6-clock.

5. Give the 7th row of the multiplication table for a 31-clock.