Equivalence Relations

A relation $\sim$ on a set $S$ is **reflexive** if

$$a \sim a \text{ for all } a \in S.$$ 

A relation $\sim$ on a set $S$ is **symmetric** if

$$a \sim b \text{ implies } b \sim a \text{ for all } a, b \in S.$$ 

A relation $\sim$ on a set $S$ is **transitive** if

$$a \sim b \text{ and } b \sim c \text{ implies } a \sim c \text{ for all } a, b, c \in S.$$ 

A relation $\sim$ on a set $S$ is an **equivalence relation** if $\sim$ is (1) reflexive, (2) symmetric, and (3) transitive.

**Example 1.** For any set $S$, $a \sim b$ means $a = b$.

**Example 2.** $S = \mathbb{Z}$, $n$ is a fixed positive integer,

$$a \sim b \text{ means } a \equiv b \pmod{n}.$$ 

**Example 3.** $S = \mathbb{R}$,

$$x \sim y \text{ means } x - y \text{ is an integer}.$$ 

**Example 4.** Consider the set of all triangles in $\mathbb{R}^2$. For any two triangles $T$ and $S$ in the plane, define $T \sim S$ to mean that $T$ is congruent to $S$ in the usual geometric sense.

Which of the following are equivalence relations? If not an equivalence relation, which of the three axioms (reflexivity, symmetry, transitivity) hold?

**Example 5.** In the set $\mathbb{Z}^+$, let $x \sim y$ mean $x$ divides $y$.

**Example 6.** $S = \mathbb{Z}$,

$$a \sim b \text{ means } (a, b) = 1.$$
Example 7. In the set \( \mathbb{R} \), let \( x \sim y \) mean \( x - y \) is a rational.

Example 8. In the set \( \mathbb{R} \), let \( x \sim y \) mean \( |x| = |y| \).

Example 9. In the set \( \mathbb{R} \), let \( x \sim y \) mean \( x \leq y \).

Example 10. In the set \( \mathbb{R} \), let \( x \sim y \) mean \( |x - y| \leq 1 \).

Example 11. In the set \( X \), let \( x \sim y \) hold for all \( x, y \in X \).

Example 12. In the set \( \mathbb{Z} \times \mathbb{Z} - \{0\} \), let \((a, b) \sim (c, d)\) mean \( ad = bc \).

Example 13. In the set \( \mathbb{R}^2 \), let \( P \sim Q \) mean that points \( P \) and \( Q \) lie on the same line through the origin.

Example 14. In the set \( \mathbb{R}^2 \), let \( P \sim Q \) mean that points \( P \) and \( Q \) lie on the same vertical line.

Example 15. In the set \( \mathbb{R}^2 \), let \( P \sim Q \) mean that points \( P \) and \( Q \) lie on the same line.

Example 16. In the set \( \mathbb{R}^2 \), let \((a, b) \sim (c, d)\) mean \( a^2 + b^2 = c^2 + d^2 \).

Example 17. For the set of all triangles in \( \mathbb{R}^2 \), let \( T \sim S \) mean that \( T \) is similar to \( S \) in the usual geometric sense.

Example 18. For the set of all subsets of \( \mathbb{R} \), let \( A \sim B \) mean that \( A \cap B = \emptyset \).

Example 19. For the set of all subsets of \( \mathbb{R} \), let \( A \sim B \) mean that \( A \subseteq B \).

**Proposition.** Suppose \( \sim \) is a transitive relation on a set \( S \). If \( x_1 \sim x_2, x_2 \sim x_3, \ldots, x_{n-1} \sim x_n \), then \( x_1 \sim x_n \).

Example 20. For the set of all people on earth, let \( P_1 \sim P_2 \) mean that \( P_1 \) looks like \( P_2 \).

Example 21. For the set of all people on earth, let \( P_1 \sim P_2 \) mean that \( P_1 \) has the same biological parents as \( P_2 \).

Example 22. For the set of all functions \( f : \mathbb{R} \to \mathbb{R} \), let \( f \sim g \) mean that \( f \) and \( g \) differ by a constant, that is, there is some constant \( c \) such that \( f(x) = g(x) + c \) for all \( x \in \mathbb{R} \).
Example 23. For the set of all functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that $f$ is a constant multiple of $g$, that is, there is some constant $k$ such that $f(x) = k \cdot g(x)$ for all $x \in \mathbb{R}$.

Example 24. For the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that $f' = g'$, where $'$ denotes the derivative.

Question 25. Here is a “proof” that symmetry and transitivity imply reflexivity:

Suppose $\sim$ is symmetric and reflexive on a set $S$. By symmetry we have $x \sim y$ implies $y \sim x$ and by transitivity, we can deduce from $x \sim y$ and $y \sim x$ that $x \sim x$, thereby establishing reflexivity.

What, if anything, is wrong with this proof?

Example 26. For the set of all subsets of $\mathbb{R}$, let $A \sim B$ mean that $A \subsetneq B$ or $B \subsetneq A$. Show that $\sim$ is symmetric and transitive. Is it reflexive?

Example 27. Given a function $f : X \to Y$ and define $a \sim b$ to mean $f(a) = f(b)$.

Equivalence Classes

If $\sim$ is an equivalence relation on a set $S$, then define the equivalence class $[a]$ to be the set $\{ b \in S : a \sim b \}$.

Theorem. Let $\sim$ be an equivalence relation on the set $S$. Then for any two equivalence classes $[a]$ and $[b]$,

$$
\begin{align*}
[a] &= [b] \quad \text{if } a \sim b \\
[a] \cap [b] &= \emptyset \quad \text{if } a \not\sim b
\end{align*}
$$

In particular, if we define $S/\sim$, the factor set of the equivalence relation $\sim$, to be

$$
S/\sim \overset{\text{def}}{=} \{ [a] : a \in S \},
$$

then the theorem above states that any two equivalence classes are either identical or disjoint. Consequently, $S$ can be partitioned into disjoint equivalence classes in $S/\sim$.

For example, $\mathbb{Z}$ is the disjoint union of the equivalence classes $[0]_3$, $[1]_3$, and $[2]_3$.

Exercise. Describe in as simple way as possible the equivalence classes of the preceding examples which turned out to be equivalence relations.
**Theorem (from book).** If \( f : S \to T \) is any function, let \( \sim \) be the equivalence relation defined on \( S \) be letting \( x_1 \sim x_2 \) if \( f(x_1) = f(x_2) \), for all \( x_1, x_2 \in S \). Then there is a one–to–one correspondence between the elements of the image \( f(S) \) of \( S \) under \( f \) and the equivalence classes in the factor set \( S/\sim \).

**Example.** [The last name function.] Let \( \mathcal{P} \) denote the set of people on earth and let \( \mathcal{N} \) denote the set of all possible names. Let \( L : \mathcal{P} \to \mathcal{N} \) be the function defined by \( L(P) = \) the last name of person \( P \). (We have to restrict people with one name, such as Cher, from our domain \( \mathcal{P} \).) We will get lots of last names, such as Smith, Jones, and Pickleheimer. Two people are related by \( \sim \) if they have the same last name. The above theorem states there is a one–to–one correspondence:

<table>
<thead>
<tr>
<th>Name</th>
<th>Equivalence Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smith</td>
<td>{people named Smith}</td>
</tr>
<tr>
<td>Jones</td>
<td>{people named Jones}</td>
</tr>
<tr>
<td>Pickleheimer</td>
<td>{people named Pickleheimer}</td>
</tr>
</tbody>
</table>

**Example.** [The remainder function mod 3.] Let \( R : \mathbb{Z} \to \{0,1,2\} \) be the function defined by \( R(n) = \) the remainder on dividing \( n \) by 3. The above theorem states there is a one–to–one correspondence:

<table>
<thead>
<tr>
<th>Remainder</th>
<th>Equivalence Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{\ldots, -6, -3, 0, 3, 6, 9, \ldots}</td>
</tr>
<tr>
<td>1</td>
<td>{\ldots, -5, -2, 1, 4, 7, 10, \ldots}</td>
</tr>
<tr>
<td>2</td>
<td>{\ldots, -4, -1, 2, 5, 8, 11, \ldots}</td>
</tr>
</tbody>
</table>

That is, each remainder \( r = 0, 1, 2 \) is associated with the congruence class \([r]_3\).