AN ELEMENTARY APPROACH TO PRIMES

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Primes and Composites

A positive integer greater than 1 is **prime** if its only divisors are 1 and itself.

The List of Primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, etc

But not $4 = 2 \times 2$, $6 = 2 \times 3$, etc.

A positive integer greater than 1 which factors is called a **composite** number.

The number 1 is called a **unit**. It is neither prime nor composite.

Question: What are the primes good for?

Grade School Problem: Add the fractions $\frac{1}{6} + \frac{1}{9}$

Solution: Find a common denominator. You could use $6 \times 9 = 54$, but a better way is to recognize that both 6 and 9 share the factor 3. So we can use 18 as a common denominator:

$$\frac{1}{6} + \frac{1}{9} = \frac{3}{18} + \frac{2}{18} = \frac{5}{18}$$

The largest number that divides both a and b is called **the greatest common divisor** of a and b or $\mathrm{GCD}(a, b)$.

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Examples: 1. GCF (6, 18) = 3

2. GCD (25, 35) = 5

2. GCD (100, 210) = 10

Katie's Theorem

Katie's Problem: Find the GCD (99, 100).

Fifth Grade Textbook Solution: Factor 99 and 100 into primes:

$$99 = 9 \times 11 = 3 \times 3 \times 11$$

$$100 = 10 \times 10 = 2 \times 2 \times 5 \times 5$$

Since none of the primes factors of 99 (3 and 11) overlap with the prime factors of 100 (2 and 5),

$$\mathrm{GCD}\,(99, 100) = 1$$

Katie's Dad's Solution: Any number which divides both 99 and 100 must divide their difference

$$100 - 99 = 1.$$

But the only number which divides 1 is 1 itself. So

$$\mathrm{GCD}\,(99, 100) = 1$$

Another example:

$$\mathrm{GCD}\,(17463287493, 17463287494) = 1$$

Katie's Theorem: Two consecutive numbers

$$n \text{ and } n + 1$$

have no factor (except 1) in common.

Are the primes plentiful or scarce?

Point of view: The primes are scarce.

Charise's Problem: For the first five consecutive birthdays after she was born, Mary's age exactly divided her grandfather's age. How old was Mary's grandfather when she was born?

Answer: 60

Mary's age	Gramps' age
1	61
2	62
3	63
4	64
5	65

Observe that 62 is the start of 4 consecutive composites.

Prime Deserts

Problem: Find 100 consecutive composite numbers.

Answer: Let

$$x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 100 \times 101.$$

Then the numbers

$$x + 2,\ x + 3,\ x + 4,\ \ldots,\ x + 101$$

are all composite. To see this, simply observe that

$$\begin{aligned}
&2 \text{ divides } x+2\\
&3 \text{ divides } x+3\\
&4 \text{ divides } x+4\\
&\vdots \quad \vdots \quad \vdots\\
&101 \text{ divides } x+101
\end{aligned}$$

The same method produces prime deserts whose size is as large as you like.

Conclusion: Primes are scarce.

Point of view: The primes are plentiful.

Euclid's Fact: You never run out of primes.

Reason: Start out like before. Let

$$x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times n.$$

What can we say about the number $x+1$?

Observe that every prime $\leq n$ divides x.

By Katie's Theorem, no prime $\leq n$ can divide $x+1$,
since no prime can divide two consecutive integers.

$$\begin{aligned}
&1 \times 2 \times 3 + 1 = 7\\
&1 \times 2 \times 3 \times 4 + 1 = 25 = 5^2\\
&1 \times 2 \times 3 \times 4 \times 5 + 1 = 121 = 11^2
\end{aligned}$$

In each case, 7, 5, 11 are **new** primes.

Conclusion: No matter how large n is, there is a prime number bigger than n.

This means that there are **infinitely** many primes; so they must be plentiful.

Second thought: Wait a minute, there are infinitely many powers of 10 (10, 100, 1000, 10000, etc.) but these do not seem very plentiful.

Enter Probability: The question "Are the primes plentiful or scarce?" really boils down to:

If I pick a large integer (at random) what is the probability it will be prime?

The Prime Number Theorem

Pick a number at random between 1 and some large bound n. Then the probability the number you chose is prime is approximately 1 out of $\ln(n)$, or, approximately, 2.3 times the number of digits of n.

Example: The probability a one hundred digit number chosen at random is prime is roughly

$$1/230$$

Application: RSA and many security systems and secret codes require large primes around 100 digits long.