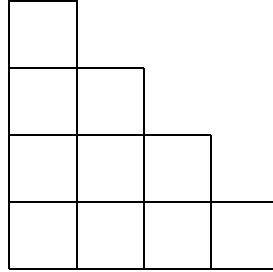


## PARITY ARGUMENTS

Can you cover the following diagram with 5 dominoes?



Hint: Think of the diagram as lying on a checkerboard and count the number of black and white squares.

## EVEN VERSUS ODD PERMUTATIONS

**Theorem.** Every permutation can be written uniquely (up to order) as a product of disjoint cycles.

**Example.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 8 & 2 & 9 & 1 & 7 & 6 & 10 & 3 \end{pmatrix} = (1, 5, 9, 10, 3, 8, 6) (2, 4) (7),$$

a product of three cycles of lengths 7, 2, and 1 respectively.

For a permutation  $\pi$ , let  $N(\pi)$  denote the number of distinct cycles of  $\pi$ , where we always include those cycles of length 1.

**Theorem.** Every cycle can be written as a product of transpositions.

Proof.

$$(a_1, a_2, a_3, \dots, a_n) = (a_1, a_n) (a_1, a_{n-1}) \dots (a_1, a_4) (a_1, a_3) (a_1, a_2).$$

**Example.**

$$(1, 5, 9, 10, 3, 8, 6) = (1, 6) (1, 8) (1, 3) (1, 10) (1, 9) (1, 5).$$

Note that  $N(id) = n$ .

**Theorem.** Let  $\tau = (a, b)$  be a transposition and  $\pi$  be a permutation. Then

$$N((a, b)\pi) = \begin{cases} N(\pi) + 1 & \text{if } a \text{ and } b \text{ belong to the same cycle of } \pi \\ N(\pi) - 1 & \text{if } a \text{ and } b \text{ belong to different cycles of } \pi. \end{cases}$$

Proof. If  $a$  and  $b$  belong to one cycle, then

$$(a, b)(a, c_1, c_2, c_3, \dots, c_k, b, d_1, d_2, \dots, d_\ell) = (a, c_1, c_2, c_3, \dots, c_k)(b, d_1, d_2, \dots, d_\ell),$$

thus increasing the number of disjoint cycles by 1. If  $a$  and  $b$  belong to different cycles, then

$$(a, b)(a, c_1, c_2, c_3, \dots, c_k)(b, d_1, d_2, \dots, d_\ell) = (a, c_1, c_2, c_3, \dots, c_k, b, d_1, d_2, \dots, d_\ell)$$

thus decreasing the number of disjoint cycles by 1.

It follows from the previous theorem that

**Theorem.** Let  $\tau_1, \tau_2, \tau_3, \dots, \tau_m$  be  $m$  transpositions and let  $\pi$  be a permutation. Then

$$N(\tau_1\tau_2\cdots\tau_m\pi) \equiv N(\pi) + m \pmod{2}.$$

There are many ways to write a permutation as a product of transpositions. In  $_4$ , for example, the permutation  $(1, 2, 3, 4)$  can be written as  $(1, 4)(1, 3)(1, 2)$  or as  $(3, 4)(2, 3)(1, 2)(2, 4)(1, 3)$ . In general, suppose a permutation  $\pi$  can be written as a product of transpositions in two different ways:

$$\pi = \tau_1\tau_2\tau_3\cdots\tau_m$$

and

$$\pi = \sigma_1\sigma_2\sigma_3\cdots\sigma_\ell.$$

Since  $N(id) = n$ , it follows from the previous theorem that

$$N(\pi) \equiv m + n \pmod{2} \quad \text{and} \quad N(\pi) \equiv \ell + n \pmod{2}.$$

Consequently  $m \equiv \ell \pmod{2}$ . This proves

**Theorem.** A given permutation is either a product of an even number of transpositions or a product of an odd number of transpositions, but never both.

A permutation is **even** if it is expressible as a product of an even number of transpositions, and **odd** if it is a product of an odd number of transpositions.

**Example:** In  $S_3$ , the even permutations are

$$id \quad (1, 2, 3) = (1, 3)(1, 2) \quad (1, 3, 2) = (1, 2)(2, 3)$$

and the odd permutations are

$$(1, 2) \quad (1, 3) \quad (2, 3).$$

### The Fifteen Puzzle

Consider the following 4 by 4 square, with the numbers 1–15 placed in the boxes:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The 15 blocks are free to slide within the large square, but they cannot be removed. By a sequence of such moves, can you ever obtain the pattern

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	