

## The $x^2 + 1$ Worksheet

Our goal is to answer the question: for which primes can you find solutions to the congruence  $x^2 + 1 \equiv 0 \pmod{p}$ .

(a) The table below lists the factorization of the first 50 numbers  $x^2 + 1$ ,  $1 \leq x \leq 50$ , into primes.

$x$	$x^2 + 1$	factors
1	2	2
2	5	5
3	10	$2 \cdot 5$
4	17	17
5	26	$2 \cdot 13$
6	37	37
7	50	$2 \cdot 5^2$
8	65	$5 \cdot 13$
9	82	$2 \cdot 41$
10	101	101

$x$	$x^2 + 1$	factors
11	122	$2 \cdot 61$
12	145	$5 \cdot 29$
13	170	$2 \cdot 5 \cdot 17$
14	197	197
15	226	$2 \cdot 113$
16		
17		
18		
19		
20	401	401

$x$	$x^2 + 1$	factors
21	442	$2 \cdot 13 \cdot 17$
22		
23		
24		
25		
26		
27		
28		
29		
30	901	$17 \cdot 53$

$x$	$x^2 + 1$	factors
31	962	$2 \cdot 13 \cdot 37$
32		
33		
34		
35		
36		
37		
38		
39		
40	1601	1601

$x$	$x^2 + 1$	factors
41	1682	$2 \cdot 29^2$
42		
43		
44		
45		
46		
47		
48		
49		
50	2501	$41 \cdot 61$

Complete the table for the missing entries. Note that in making the list, it is very helpful to know when a number, such as 101 or 197 is prime. I will give you a table of primes up to 5039. A slightly larger table up to 10007 is available on the Math 420 website.

(b) For each prime in the table below, fill in all values of  $x$  (up to  $x = 50$ ) for which  $p$  divides  $x^2 + 1$ .

$p$	$x : p \text{ divides } x^2 + 1$
2	
3	
5	
7	
11	
13	
17	
19	
23	
29	
31	
37	
41	

$p$	$x : p \text{ divides } x^2 + 1$
43	
47	
53	
61	
67	
71	
73	
79	
83	
89	
91	
97	
101	

(c) What can you say about the numbers in each row of the table?

(d) Which primes appear in the list? Which primes do not? Can you think of a general rule which determines whether a prime is in the list or not?