The Pails of Water Problem

You have a 5 and a 7 quart pail. How can you measure exactly 1 quart of water, by pouring water back and forth between the two pails? You are allowed to fill and empty each pail as many times as needed. [Try to solve this problem before reading further!] Here is one solution:

5-quart	**7-quart**	**Explanation**
0	0	both pails start out empty
5	0	fill 5-quart pail
0	5	empty 5-quart pail into 7
5	5	fill 5-quart pail a second time
3	7	pour 2 quarts from 5-quart pail to fill up 7
3	0	empty 7-quart pail
0	3	transfer 3 quarts from 5-quart pail into 7
5	3	fill 5-quart pail a third time
1	7	pour 4 quarts from 5-quart pail to fill up 7
1	0	empty 7-quart pail a second time.

This leaves exactly one quart in the 5-quart pail. Notice that we filled the 5-quart pail *three* times and emptied the 7-quart pail *twice*. A little thought convinces us that the method given above can be written more succinctly as:

$$3 \times 5 - 2 \times 7 = 1.$$

Now suppose you wish to obtain one quart with a 7-quart and a 19-quart pail. [Try this before reading further!] The secret is to fill the 19-quart pail two times from the 7-quart pail, thereby effectively converting the 19-quart pail into a 5-quart pail. The associated equation is $5 = 19 - 2 \times 7$. But we know how to solve the problem with quart sizes of 7 and 5. So we can use our previous solution for 5 and 7 to get a solution for 7 and 19. Here's how it works:

$$\begin{aligned}
1 &= 3 \times 5 - 2 \times 7 && \text{solution for 5 and 7} \\
&= 3 \times (19 - 2 \times 7) - 2 \times 7 && \text{substituting } 5 = 19 - 2 \times 7 \\
&= 3 \times 19 - 6 \times 7 - 2 \times 7 && \text{expand the parentheses} \\
&= 3 \times 19 - 8 \times 7 && \text{gather the 7's together}
\end{aligned}$$

The solution is to fill the 19-quart pair three times, a total of 57 quarts, and empty it into the 7-quart pail eight times, a total of 56 quarts. The difference is precisely 1 quart.

Let's solve the Pails of Water Problem (POW) with the numbers 100 and 77. First, we divide 77 into 100. Since 77 goes into 100 one time, with remainder 23, we write

$$23 = 100 - 77$$

and record this equation for later use. Now we have reduced the POW problem to the numbers 77 and 23. But these numbers are still too big; it is not obvious how to get 1 from the numbers 77 and 23. So we reduce again. The smaller number 23 goes into 77 three times, with remainder 8. Write this as

$$8 = 77 - 3(23).$$

Can we solve POW with 23 and 8? Yes! $3 \times 8 - 1 \times 23 = 1$. Now we substitute the previous "remainder" equations to work our way back to the original numbers 100 and 77:

$$\begin{aligned}
1 &= 3(8) - 23 && \text{solution with 8 and 23} \\
&= 3(77 - 3(23)) - 23 && \text{substitute } 8 = 77 - 3(23) \\
&= 3(77) - 9(23) - 23 && \text{expand the first parentheses} \\
&= 3(77) - 10(23) && \text{gather the 23's together} \\
&= 3(77) - 10(100 - 77) && \text{substitute } 23 = 100 - 77 \\
&= 3(77) - 10(100) + 10(77) && \text{expand the second parentheses} \\
&= 13(77) - 10(100) && \text{gather the 77's together.}
\end{aligned}$$

We can easily check our solution $13 \times 77 - 10 \times 100 = 1$ with a calculator: $13 \times 77 = 1001$ and we know (without a calculator) that $10 \times 100 = 1000$.

In general, suppose you want to solve the POW problem with two numbers, a and b. Let's say that a is smaller than b. If you can't think of the solution right away because the numbers are too large, reduce the problem to two smaller numbers, by dividing the smaller number a into b, getting a quotient q and a remainder r. Write this as

$$r = b - q \times a.$$

Record this equation as you will need it later. The original POW problem a and b *reduces* to solving POW with the numbers r and a. You can write this reduction as $(a, b) \to (r, a)$. If the numbers r and a are still too big, reduce again, by dividing r into a. Eventually you will reach two numbers that you can do in your head. Starting with this "easy" solution, carefully work backwards, substituting the "remainder" equations you previously recorded, until you reach a solution using the original numbers a and b.

The method described in this section for solving the pails of water problem is called the *Euclidean algorithm*. An algorithm is a set of steps for completing some task. (No, the word is not named after Bill Clinton's vice president.) A recipe from a cookbook is a good example of an algorithm. This particular procedure dates back over two thousand years ago to the African mathematician Euclid, who also founded plane geometry.

The Matrix Method

Solve POW(100,77) one more time. What we are looking for is an equation of the form

$$ax + by = 1$$

which has the solution $x = 100$ and $y = 77$. This looks like a linear algebra problem, where we know that the final matrix in a system of row reductions is

$$\begin{bmatrix} 1 & 0 & | & 100 \\ 0 & 1 & | & 77 \end{bmatrix}$$

and we need to work backwards to construct the original equation $ax + by = 1$. We do this by adding multiples of the last row to the next-to-the-last row. The division algorithm tells us what multiple to use. For example, from $100 = 1 \cdot 77 + 23$, it is clear that if we add -1 times 77 to 100, we will be left with 23 in the last column. Continuing gives the following sequence of rows:

$$\begin{pmatrix} 1 & 0 & | & 100 \\ 0 & 1 & | & 77 \\ 1 & -1 & | & 23 \\ -3 & 4 & | & 8 \\ 7 & -9 & | & 7 \\ -10 & 13 & | & 1 \end{pmatrix} \qquad \begin{array}{l} x = 100 \\ y = 77 \\ 100 + (-1) \cdot 77 = 23 \\ 77 + (-3) \cdot 23 = 8 \\ 23 + (-2) \cdot 8 = 7 \\ 8 + (-1) \cdot 7 = 1 \end{array}$$

Questions:

1. You cannot obtain 1 quart of water with 6 and 10 quart pails. Why not? What is the smallest amount you can obtain?

2. In general, given positive integers a and b, what do you think is the smallest amount of water you can obtain from pails of size a and b?

3. Solve POW in your head: (a) 10 and 19; (b) 7 and 9; (c) 11 and 8

4. Another solution with 5 and 7 is: $3 \times 7 - 4 \times 5 = 1$. Show how to get this solution from the solution $3 \times 5 - 2 \times 7 = 1$. Hint: add and subtract 5×7.