Shuffles

The "10-card shuffle" (Note that I am not playing with a full deck. Ha, ha.) is performed as follows:

The cards 1, 2, 3, . . . , 9, 10 are in order, with 1 being the top card and 10 on the bottom.

Cut the cards in the middle, making two piles: Pile A contains the cards 1 – 5 and Pile B contains the cards 6 – 10.

Now interleave these two decks, alternating between Pile A and Pile B, first laying down the 5 at the bottom of Pile A, followed by the 10 at the bottom of Pile B, etcetera, ending with the 6 from Pile B on top.

	Pile A	Pile B	
2nd	1	6	top
4th	2	7	3rd
6th	3	8	5th
8th	4	9	7th
last	5	10	9th

The new deck, from top to bottom is

1	2	3	4	5	6	7	8	9	10
6	1	7	2	8	3	9	4	10	5

If we shuffle, following this method, a second time, we get

1	2	3	4	5	6	7	8	9	10
3	6	9	1	4	7	10	2	5	8

A third shuffle gives

1	2	3	4	5	6	7	8	9	10
7	3	10	6	2	9	5	1	8	4

A fourth shuffle gives

1	2	3	4	5	6	7	8	9	10
9	7	5	3	1	10	8	6	4	2

A fifth shuffle gives

1	2	3	4	5	6	7	8	9	10
10	9	8	7	6	5	4	3	2	1

A sixth shuffle gives

1	2	3	4	5	6	7	8	9	10
5	10	4	9	3	8	2	7	1	6

A seventh shuffle gives

1	2	3	4	5	6	7	8	9	10
8	5	2	10	7	4	1	9	6	3

An eighth shuffle gives

1	2	3	4	5	6	7	8	9	10
4	8	1	5	9	2	6	10	3	7

An ninth shuffle gives

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9

An tenth shuffle gives

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10

Construct the multiplication table for $\mathbb{Z}_{11}^{\times}$. Do you recognize these permutations?

A simpler way to write permutations is to track what each number maps to, starting with 1, until you eventually get back to 1.

$$\boxed{1 \to 6 \to 3 \to 7 \to 9 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1}$$

This cycle is usually written with commas instead of the arrows:

$$\boxed{\sigma = (1, 6, 3, 7, 9, 10, 5, 8, 4, 2)}$$

Notice that we don't write 1 at the end, as it is understood from the diagram that the last element, 2, points to the first element of the cycle, 1.

Compute the order of 6 in $\mathbb{Z}_{11}^{\times}$. What numbers in the ten shuffles do you see?

The permutation σ^2 is computed by applying the permutation σ to itself

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \sigma(\sigma(1)) & \sigma(\sigma(2)) & \sigma(\sigma(3)) & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma(\sigma(9)) & \sigma(\sigma(10)) \end{pmatrix}$$

or

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 \end{pmatrix},$$

the "second" shuffle above.

Four ways to view permutations:

(1) as a shuffle of the numbers 1, 2, 3, $\cdots$, n
(2) using the "matrix" notation $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}$
(3) possibly as a row of a multiplication table mod a prime
(4) using the cycle notation $(i_1, i_2, i_3, \cdots, i_j)$

Do you think all permutations can be written as a multiplication mod n, for some n?

What is the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix} \ ?$$

Why, when we compose permutations, must we work **from right to left**?

Would it make a difference in general, if we worked left to right?

Would it make a difference in computing the powers σ^k?

What is the identity permutation?

What is the inverse permutation σ^{-1}? What does this mean in terms of a shuffle?

If we take larger and larger values of k, will σ^k eventually become the identity permutation? The smallest positive k for which $\sigma^k = 1_n$ is called the **order** of σ. This questions asks: does every permutation have an order?

Exercise: What happens if you shuffle a regular deck of 52 cards by interleaving two piles (1–26) and (27–52) as we did for the ten card deck.

The following chart might help you to compute the cycle(s):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 27 & 1 & 28 & 2 & 29 & 3 & 30 & 4 & 31 & 5 & 32 & 6 & 33 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\ 7 & 34 & 8 & 35 & 9 & 36 & 19 & 37 & 11 & 38 & 12 & 39 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 40 & 14 & 41 & 15 & 42 & 16 & 43 & 17 & 44 & 18 & 45 & 19 & 46 \end{pmatrix}$$

$$\begin{pmatrix} 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 \\ 20 & 47 & 21 & 48 & 22 & 49 & 23 & 50 & 24 & 51 & 25 & 52 & 26 \end{pmatrix}$$