1. Math 423

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2. Lemma

Suppose V is a vector space with basis $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_n\}$.

Then any set of $n+1$ vectors is linearly dependent.

The proof uses induction on n.

3. Dimension

The Lemma easily proves the following theorem:

Theorem. Suppose V is a vector space with basis $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_n\}$.

Then every basis of V also consists of n vectors.

n is called the **dimension** of V.

4. Induction Proofs

To prove a statement $P(n)$ by induction requires two steps:

Step 1. Show that $P(1)$ is true.

Step 2. Show that if $P(n)$ is true, then $P(n+1)$ is also true.

Equivalently, we can show that $P(n-1) \implies P(n)$.

5. Case $n = 1$

Given a set of two vectors $\{\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2\}$.

Since $\{\underline{\mathbf{v}}_1\}$ is a basis, we can write

$\underline{\mathbf{w}}_1 = a\underline{\mathbf{v}}_1$ and

$\underline{\mathbf{w}}_2 = b\underline{\mathbf{v}}_1$

for appropriate scalar constants a and b.

If either $a =$ or $b = 0$,

then $\underline{\mathbf{w}}_1$ or $\underline{\mathbf{w}}_1$, respectively, is $\underline{\mathbf{0}}$, the zero vector.

Clearly, $\{\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2\}$ is linearly dependent in this case.

6. Case $n = 1$ continued

Given a set of two vectors $\{\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2\}$.

$\underline{\mathbf{w}}_1 = a\underline{\mathbf{v}}_1$ and $\underline{\mathbf{w}}_2 = b\underline{\mathbf{v}}_1$.

where both a and b are non-zero.

In this case,

$b\underline{\mathbf{w}}_1 - a\underline{\mathbf{w}}_2 = ba\underline{\mathbf{v}}_1 - ab\underline{\mathbf{v}}_1 = \underline{\mathbf{0}}$, the zero vector.

Since a and b are not both 0, this equation proves that

$\{\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2\}$ is linearly dependent.

7. Induction Hypothesis

Suppose $n > 1$ and the lemma is true for $n - 1$.

Given a basis $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_n\}$ for vector space V.

Let W be the vector space spanned by $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_{n-1}\}$.

Then $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_{n-1}\}$ is a basis for W.

Clearly $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_{n-1}\}$ span W?.

Why is $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_{n-1}\}$ linearly independent?

8. Truncate vectors

Let $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_{n+1}$ be $n+1$ vectors in V.

We must show that $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_{n+1}$ is linearly dependent.

Since $\{\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_n\}$ is a basis for V,

there are scalars $a_{i,j}$, $1 \leq i \leq n+1$, $1 \leq j \leq n$, such that

$\underline{\mathbf{w}}_i = a_{i,1}\underline{\mathbf{v}}_1 + \cdots + a_{i,n}\underline{\mathbf{v}}_n \quad i = 1, 2, \ldots, n+1.$

Set

$\underline{\mathbf{z}}_i = a_{i,1}\underline{\mathbf{v}}_1 + \cdots + a_{i,n-1}\underline{\mathbf{v}}_{n-1} \quad i = 1, \ldots, n+1.$

Note that

$\underline{\mathbf{w}}_i = \underline{\mathbf{z}}_i + a_{i,n}\underline{\mathbf{v}}_n$

9. Use induction hypothesis

Since $\underline{\mathbf{v}}_1, \ldots, \underline{\mathbf{v}}_{n-1}$ is a basis for W, by the induction hypothesis,

both $\underline{\mathbf{z}}_1, \ldots, \underline{\mathbf{z}}_n$ and $\underline{\mathbf{z}}_1, \ldots, \underline{\mathbf{z}}_{n-1}, \underline{\mathbf{z}}_{n+1}$ are linearly dependent.

Therefore, without loss of generality, we may write

$\underline{\mathbf{z}}_n = b_1\underline{\mathbf{z}}_1 + \cdots + b_{n-1}\underline{\mathbf{z}}_{n-1}$

and

$\underline{\mathbf{z}}_{n+1} = c_1\underline{\mathbf{z}}_1 + \cdots + c_{n-1}\underline{\mathbf{z}}_{n-1}$

for appropriate scalars b_i and c_i.

Question: Why do we say "without loss of generality"?

10. Set up two equations

We have

$\underline{\mathbf{0}} = b_1\underline{\mathbf{z}}_1 + \cdots + b_{n-1}\underline{\mathbf{z}}_{n-1} - \underline{\mathbf{z}}_n$

$= b_1(\underline{\mathbf{w}}_1 - a_{(1,n)}\underline{\mathbf{v}}_n) + \cdots + b_{n-1}(\underline{\mathbf{w}}_{n-1} - a_{(n-1,n)}\underline{\mathbf{v}}_n) - (\underline{\mathbf{w}}_n - a_{(n,n)}\underline{\mathbf{v}}_n)$

$= b_1\underline{\mathbf{w}}_1 + \cdots + b_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_n - b\underline{\mathbf{v}}_n$

for some constant b.

So $b_1\underline{\mathbf{w}}_1 + \cdots + b_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_n = b\underline{\mathbf{v}}_n$

Similarly,

$c_1\underline{\mathbf{w}}_1 + \cdots + c_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_{n+1} = c\underline{\mathbf{v}}_n$

for some choice c.

11. Finish the proof

We have

$b_1\underline{\mathbf{w}}_1 + \cdots + b_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_n = b\underline{\mathbf{v}}_n$

and

$c_1\underline{\mathbf{w}}_1 + \cdots + c_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_{n+1} = c\underline{\mathbf{v}}_n$

If $b = 0$, we are done, since now $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_n$ are linearly dependent.

If $b \neq 0$, multiply the first equation by c and multiply the second equation by b to get

$c\big(b_1\underline{\mathbf{w}}_1 + \cdots + b_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_n\big) = b\big(c_1\underline{\mathbf{w}}_1 + \cdots + c_{n-1}\underline{\mathbf{w}}_{n-1} - \underline{\mathbf{w}}_{n+1}\big),$

which shows that $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_{n+1}$ are linearly dependent.