Math Club Talk

Galaxies of Primes
Primes and Composites

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The List of Primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, etc
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A positive integer greater than 1 which factors is called a **composite** number.

The number 1 is called a **unit**. It is neither prime nor composite.
What are the primes good for?

Grade School Problem:
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Add the fractions $\frac{1}{6} + \frac{1}{9}$
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Solution: Find a common denominator.
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Solution: Find a common denominator.
You could use $6 \times 9 = 54$, but a better way is to recognize that both 6 and 9 share the factor 3.
What are the primes good for?

Grade School Problem:
Add the fractions $\frac{1}{6} + \frac{1}{9}$

Solution: Find a common denominator. You could use $6 \times 9 = 54$, but a better way is to recognize that both 6 and 9 share the factor 3. So we can use 18 as a common denominator:

$$\frac{1}{6} + \frac{1}{9} = \frac{3}{18} + \frac{2}{18} = \frac{5}{18}$$
GCD

The largest number that divides both $a$ and $b$ is called the greatest common factor of $a$ and $b$ or $\text{GCD}(a,b)$.

Examples:
1. $\text{GCF}(6, 18) = 3$
2. $\text{GCF}(25, 35) = 5$
3. $\text{GCF}(100, 210) = 10$
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The largest number that divides both \( a \) and \( b \) is called the greatest common factor of \( a \) and \( b \) or \( \text{GCD}(a, b) \).

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Katie’s Theorem

Katie’s Problem: Find the GCD (99, 100).

Fifth Grade Textbook Solution:

Factor 99 and 100 into primes:

\[ 99 = 3 \times 33 = 3 \times 11 \]
\[ 100 = 2 \times 50 = 2 \times 5 \times 5 \]

Since none of the prime factors of 99 (3 and 11) overlap with the prime factors of 100 (2 and 5),

\[ \text{GCD}(99, 100) = 1 \]
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Fifth Grade Textbook Solution: Factor 99 and 100 into primes:
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Katie’s Dad’s Solution

Any number which divides both 99 and 100 must divide their difference

\[ 100 - 99 = 1. \]
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But the only number which divides 1 is 1 itself.
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Another example:
\( \text{GCD}(17463287493, 17463287494) = \)
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Two consecutive numbers

\[ n \text{ and } n + 1 \]

have no factor (except 1) in common.
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Are the primes plentiful or scarce?
The primes are scarce

Typical Algebra Word Problem: Lucy and Lottie Hill are 90 years old. Mary Jane, on the other hand, is half again as old as she was when she was half again as old as she was when she lacked five years of being half as old as she was on her last birthday.
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Hint: Explain the first sentence about Lucy and Lottie Hill.
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Charise’s Problem: For the first five consecutive birthdays after she was born, Mary’s age exactly divided her grandfather’s age.
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Hint: Explain the first sentence about Lucy and Lottie Hill.
Charise’s Problem: For the first five consecutive birthdays after she was born, Mary’s age exactly divided her grandfather’s age. How old was Mary’s grandfather when she was born?
Answer to Charise’s Problem

Grandpa was 60
Answer to Charise’s Problem

Grandpa was 60
  Mary’s age  Gramps’ age

Observe that 62 is the start of 4 consecutive composites.
Answer to Charise’s Problem

Grandpa was 60

Mary’s age  Gramps’ age

1
2
3
4
5
## Answer to Charise’s Problem

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Prime Deserts

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\[ x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 100 \times 101 \]
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Then the numbers \( x + 2, x + 3, x + 4, \ldots, x + 101 \) are all composite.
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Then the numbers \( x + 2, x + 3, x + 4, \ldots, x + 101 \) are all composite.
To see this, simply observe that
2 divides \( x + 2 \)
3 divides \( x + 3 \)
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The same method produces prime deserts whose size is as large as you like.

**Conclusion:** Primes are scarce.
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**Conclusion:** Primes are scarce.
The primes are plentiful

Euclid’s Theorem: You never run out of primes.
Reason: Start out like before.
The primes are plentiful

**Euclid’s Theorem:** You never run out of primes.

**Reason:** Start out like before.

Let

\[ x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times n. \]
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What can we say about the number \( x + 1 \)?
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What can we say about the number \( x + 1 \)?
Observe that every prime \( \leq n \) divides \( x \).
By Katie’s Theorem, no prime \( \leq n \) can divide \( x + 1 \),
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What can we say about the number \( x + 1 \)?
Observe that every prime \( \leq n \) divides \( x \).
By Katie’s Theorem, no prime \( \leq n \) can divide \( x + 1 \),
since no prime can divide two consecutive integers.
Thus the prime divisors of \( x + 1 \) are all new primes,
that is, they are all \( > n \).
Some Examples

$1 \times 2 \times 3 + 1 =$

In each case, 7, 5, 11 are new primes.

Conclusion: No matter how large $n$ is, there is a prime number bigger than $n$. This means that there are infinitely many primes; so they must be plentiful.
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$1 \times 2 \times 3 + 1 = 7$
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\[ 1 \times 2 \times 3 \times 4 + 1 = 25 = 5^2 \]

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This means that there are infinitely many primes; so they must be plentiful.
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Wait a minute, there are infinitely many powers of 10:

10, 100, 1000, 10000, etc.
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10, 100, 1000, 10000, etc.

but these do not seem very plentiful.

Enter Probability: The question “Are the primes plentiful or scarce?” really boils down to:
If I pick a large integer (at random) what is the probability it will be prime?
The Prime Number Theorem

Pick a number at random between 1 and some large bound $n$. 

Example: The probability a one hundred digit number chosen at random is prime is roughly $\frac{1}{230}$. 

Application: RSA and many security systems and secret codes require large primes around 100 digits long. The Prime Number Theorem suggests how many one hundred digit numbers a computer program will need to check before it finds a prime.
The Prime Number Theorem

Pick a number at random between 1 and some large bound $n$. Then the probability the number you chose is prime is approximately 1 out of 2.3 times the number of digits of $n$. 

Example: The probability a one hundred digit number chosen at random is prime is roughly $1/2^{100} = 2^{30}$.

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**Application:** RSA and many security systems and secret codes require large primes around 100 digits long.

The Prime Number Theorem suggests how many one hundred digit numbers a computer program will need to check before it finds a prime.
Twin Primes

Notice that primes sometimes come in pairs, two apart:

3 and 5
5 and 7
11 and 13
17 and 19

This last example has more than 51,000 digits!

Such numbers are called twin primes.
Twin Primes

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In the universe of numbers, they are like twin stars, a constellation consisting of exactly two nearby stars.
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$33218925 \times 2^{169690} - 1$ and $33218925 \times 2^{169690} + 1$
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This last example has more than 51,000 digits!
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Notice that primes sometimes come in pairs, two apart:
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Such numbers are called \textbf{twin primes}.
In the universe of numbers, they are like twin stars, a constellation consisting of exactly two nearby stars.
Prime Constellations

A prime constellation consists of a fixed number of primes, where each prime is a set distance apart from each other.

Question: Can you have three primes, each two units apart: $p$, $p + 2$, $p + 4$?

Hint: Write down three consecutive odd integers, such as 23, 25, 27. One of them must be divisible by 3. Do you see why?

Question: What about $p$, $p + 2$, $p + 6$?

Examples: 5, 7, 11; 17, 19, 23.
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Cluster Primes

A prime $p > 2$ is called a cluster prime if every even positive integer $\leq p - 3$ can be written as a difference of two primes which are both $\leq p$. 

The smallest non-cluster prime is 97.

\begin{align*}
88 &= 97 - 9 \\
88 &= 97 - 7 \\
88 &= 97 - 5 \\
88 &= 97 - 3 \\
88 &= 89 - 1
\end{align*}

Hence 88 is not a difference of two primes 97. So 97 is not a cluster prime.
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Galaxies of Primes

In general, if $p$ is a cluster prime, then there must be enough primes in a “small” neighborhood to the left of $p$ so that the even numbers $p - 9, p - 15, p - 21$, etc., can all be written as the difference of primes less than $p$. 
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To the left of a cluster prime we see a galaxy of primes, where the primes are packed together much like at the beginning of the list of primes 2, 3, 5, etc.
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To the left of a cluster prime we see a galaxy of primes, where the primes are packed together much like at the beginning of the list of primes 2, 3, 5, etc. One would expect that as we journey through the positive integer, the cluster primes producing these “prime galaxies” would be increasingly rare.
Density of the cluster primes

\[ C(x) = \text{cluster}; \]
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<table>
<thead>
<tr>
<th>( x )</th>
<th>( C(x) )</th>
<th>( N(x) )</th>
<th>( \frac{N(x)}{C(x)} )</th>
<th>( T(x) )</th>
</tr>
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<tbody>
<tr>
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<td>68</td>
<td>0.69</td>
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<td>( 10^4 )</td>
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<td>( 10^5 )</td>
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<td>8.47</td>
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<tr>
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<tr>
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<td>1870585220</td>
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It is reasonable to expect that among the primes, the cluster primes become increasingly rare. In spite of the initial head start of 23 consecutive cluster primes, the non-cluster primes quickly catch up, so that by the prime 2251 we have 167 cluster primes and 167 non-cluster primes. Starting with 2267, the next prime after 2251, the cluster primes begin to lag further and further behind. When we reach $10^{13}$, the non-cluster primes outnumber the cluster primes by a ratio of about 325 to 1.
Infinitely many cluster primes?

An affirmative answer would imply that
\[ p_{n+1} - p_n \leq 6 \] for infinitely many primes \( p_n \),
Infinitely many cluster primes?

An affirmative answer would imply that $p_{n+1} - p_n \leq 6$ for infinitely many primes $p_n$, which is a well-known hopeless problem.
Upper Bound for $C(x)$

We enjoy more success looking for an upper bound for $C(x)$, the number of cluster primes $\leq x$. 
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Upper Bound for $C(x)$

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Asymptotic functions

We say that two functions $f(x)$ and $g(x)$ are asymptotic if and only if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

Example:

$$7x^3 + 11x^2 + 49x$$

is asymptotic to

$$7x^3$$

because

$$\lim_{x \to \infty} \frac{7x^3 + 11x^2 + 49x}{7x^3} = \lim_{x \to \infty} \left(1 + \frac{11}{7x} + \frac{49}{7x^3}\right) = 1$$
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Primes and twin primes density

Primes.

Theorem [Prime Number Theorem]
\[ P(x) \sim x \log x \]

Twin Primes.

Theorem [Brun]
Let \( T(x) = \) the number of twin primes \( \leq x \).
Then eventually
\[ T(x) < M (\log x)^2 \]
for some constant \( M \).

Conjecture [Hardy and Littlewood]
\( T(x) \), the number of twin primes \( \leq x \) is asymptotic to \( \frac{1}{320323632} \int x^2 \, dx \log x \),
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$$1.320323632 \int_{2}^{x} \frac{dx}{(\log x)^2}.$$
Density of the cluster primes

Theorem
[Blecksmith, Erdos, and Selfridge] For every positive integer \( s \), there is a bound \( x_0 = x_0(s) \) such that if \( x > x_0 \) then
\[
C(x) < x \log x
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Conjecture
[Blecksmith, Erdos, and Selfridge] For some constant \( C \),
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C(x) \sim x e^{\log^2 x}
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Density of the cluster primes

**Theorem** [Blecksmith, Erdős, and Selfridge] For every positive integer $s$, there is a bound $x_0 = x_0(s)$ such that if $x \geq x_0$ then

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Density of the cluster primes

**Theorem** [Blecksmith, Erdős, and Selfridge] For every positive integer $s$, there is a bound $x_0 = x_0(s)$ such that if $x \geq x_0$ then

$$C'(x) < \frac{x}{(\log x)^s}.$$ 

**Conjecture** [Blecksmith, Erdős, and Selfridge] For some constant $\alpha$, $C'(x)$ is asymptotic to

$$\frac{x}{e^{\alpha(\log \log x)^2}}.$$