

MATH 360  
SAMPLE PROJECT

Locating a railway supply depot

Several factors influence the location of factories, power plants, retail stores, and other commercial buildings. The placement of machines in a production facility is similarly subject to a number of considerations. Final decisions are made by balancing the pros and cons of various options. In this project we will show that in some circumstances we can find an optimal location easily.

This is the scenario we consider:

Because of extreme weather conditions, lots of maintenance and repairs are needed for the trains which run on the trans-Siberian railway. This maintenance is carried out at a number of sidings (repair stations) which are located at various points along the track. Parts are scarce and it is not economical to stock them at all the repair facilities. Instead, it is planned to implement a “just in time” supply procedure: a Parts Distribution Center is to be built somewhere along the track. Parts will be shipped from there to the various repair facilities. The problem is to choose a location which will minimize the cost of *supplying* the parts to the sidings.

How could we possibly solve this problem? We need to translate it into mathematical terms. The first step is to decide on a way of representing the setting for the problem. Since the setting is a single railroad track, it is essentially one-dimensional, so we will represent the railroad track by the  $x$ -axis. (We must select a point to be the origin, and select our unit of measurement, but those choices won't affect our answers.) Then the sidings can be represented by points  $x_i$  ( $i = 1, 2, \dots, n$ ) on the  $x$ -axis, where  $n$  is the number of sidings. We'll likewise let the unknown  $x$  be the position of the parts center. There is a certain cost  $C(x)$  associated with siting the parts center at any point  $x$ ; the problem then becomes: Find the location  $x$  of the parts center so that the cost  $C(x)$  of supplying parts to  $x_1, \dots, x_n$  is minimized.

At one level, the problem is of a familiar type. It is a maximum/minimum problem, and we know that calculus provides a tool for solving such problems. However, what is the function  $C(x)$  to be minimized? This isn't given directly in the statement of the problem. We shall have to figure it out.

The cost of supplying parts will depend on the distance through which they must be moved. This, in turn, will depend on the number of trips to each siding  $x_i$ , and the distance from  $x$  to  $x_i$ .

Denote by  $N_i$  the number of trips to siding  $x_i$  in a year. Then the total distance involved in all the supply trips to repair station  $i$  is  $N_i|x - x_i|$ .

Now, how does this determine the cost? The statement of the problem does not say; what's a reasonable assumption? Costs primarily associated with transit (say, driver's hourly wage, or fuel cost) are roughly proportional to the distance traveled, say,  $C = kD$ . Let us assume that this is in fact how  $C(x)$  is to be computed; that is, we make explicit the assumption that there is a unit cost  $k$  such that total cost is  $k$  times the distance traveled.

Then the cost of maintaining the parts center at  $x$  is of the form  $C(x) = k D(x)$  where

$$D(x) = k\{N_1|x - x_1| + \dots + N_n|x - x_n|\}$$

is the total distance traveled. Our problem becomes, “Find the value of  $x$  which minimizes  $C(x)$ ”.

Before trying to solve this problem, we note that we have made several assumptions in setting up the model, and so there are several questions about the model that we should consider. For example, is the model a realistic one? The model depends on the distance from  $x$  to each  $x_i$ , and also on the number  $N_i$  of trips from the parts center to  $x_i$ . How can we determine  $N_i$ ? We shall return to these questions later.

At first this seems like a simple calculus problem. We know how to solve those: just take the derivative and set it to zero, right? The problem is that the function  $C(x)$  doesn't have a derivative everywhere because of the absolute value bars, so we can't just take the derivative. However, we can use theorems from calculus to help us solve the problem. Let's see what we do know about  $C(x)$ .

1. Since  $C(x)$  is just a constant multiple of the function  $D(x)$ , and since  $k > 0$ ,  $C(x)$  and  $D(x)$  will have their minima at the same point  $x$ . So it will be enough for us to find the  $x$  which minimizes  $D$ .

2.  $D(x)$  is the sum of the functions  $D_i(x) = N_i|x - x_i|$  for  $i = 1, \dots, n$ . Each of these is a continuous function. Therefore, since the sum of continuous functions is continuous,  $D(x)$  is continuous for all values of  $x$ . Since we know that  $D(x)$  is continuous, a theorem from calculus tells us that  $D(x)$  has a minimum value on  $[A, B]$ , so there really is a solution to the problem.

3.  $D_i(x) = N_i|x - x_i|$  is differentiable for all  $x$  different from  $x_i$ . For example,  $y = 5|x - 2|$  is differentiable for all values of  $x$  except  $x = 2$ . Further, the graph is a straight line, that is, the function is linear, on each of the intervals  $(-\infty, 2)$  and  $(2, \infty)$ . Since the derivative of a sum is the sum of the derivatives provided the derivative of each summand exists, we see that  $D(x)$  is differentiable at every point except  $x_1, \dots, x_n$ .

4. The minimum value of  $D(x)$  occurs at one of the following:

- at one of the ends of the interval;
- at a point in the interior of the interval where the derivative is zero;
- at a point in the interior of the interval where the derivative does not exist.

5. Since the sum of linear functions is linear, we know that  $D(x)$  is linear on each of the intervals  $[A, x_1]$ ,  $[x_1, x_2]$ ,  $\dots$ ,  $[x_n, B]$  and so the minimum value of  $D(x)$  on such an interval occurs at one of the end points. This means that the minimum value of  $D(x)$  on all of  $[A, B]$  occurs at one of the points  $A, x_1, \dots, x_n, B$ . To find an optimal location we can calculate the values of  $D(x)$  at these  $n + 2$  points and find those locations where the value is least.

The solution we have given is a qualitative one. It uses qualitative properties of the model — continuity, differentiability, etc. — to show that the problem has a solution and, in qualitative terms, where the solution may be found. But how good a solution is this? There are still questions to be asked.

For example, how many solutions are there? If there are several, how do we distinguish between them? (Maybe one is better than another for some purposes.) How efficient is the solution? That is, how much computational effort is required to find it? If there are only a few sidings, this isn't really a problem but what if many locations were involved? The solution we have given doesn't use any additional information we might have about the particular locations  $x_i$  and costs  $N_i$ . Can we get a better solution process if we do take these into account? In other words, can we *refine* the solution we have already obtained to find a better one?

Let's look at an example where there are three sidings located at 2,3, and 5 on the  $x$ -axis, and where

$$D(x) = 4|x - 2| + |x - 3| + 2|x - 5|.$$

To see what is going on, we consider the graph of  $D(x)$  on the subintervals

$$(-\infty, 2), (2, 3), (3, 5), (5, \infty)$$

of the  $x$ -axis because the absolute values change on these intervals. To the left of  $x = 2$ , for example,  $|x - 2| = 2 - x$ ,  $|x - 3| = 3 - x$ , and  $|x - 5| = 5 - x$ , so that  $D(x) = 21 - 7x$ , whose graph is a straight line with slope  $-7$ . Likewise in the the remaining intervals, we calculate  $D(x) = x + 5$ ,  $3x - 1$ , and  $7x - 21$ , respectively.

Looking at the graph then shows there is exactly one location where the minimum value occurs, namely  $x = 2$ .

To see what is going on in the general situation where there are  $n$  locations we look separately at each of the intervals

$$(-\infty, x_1), (x_1, x_2), \dots, (x_n, \infty).$$

First suppose  $x$  is in  $(-\infty, x_1)$ . Then  $|x - x_i| = x_i - x$  for each  $i$  so that

$$\begin{aligned} D(x) &= N_1|x - x_1| + \dots + N_n|x - x_n| \\ &= N_1(x_1 - x) + \dots + N_n(x_n - x) \\ &= (N_1x_1 + \dots + N_nx_n) - (N_1 + \dots + N_n)x \end{aligned}$$

which is the equation of a straight line with negative slope

$$S_0 = -(N_1 + \dots + N_n).$$

Next consider  $(x_n, \infty)$ . Here  $|x - x_i| = x - x_i$  for each  $i$  so that the situation this time is the opposite of that which we considered before. We get

$$\begin{aligned} D(x) &= N_1|x - x_1| + \dots + N_n|x - x_n| \\ &= N_1(x - x_1) + \dots + N_n(x - x_n) \\ &= (N_1 + \dots + N_n)x - (N_1x_1 + \dots + N_nx_n) \end{aligned}$$

which is the equation of a straight line with positive slope

$$S_n = N_1 + \dots + N_n.$$

It follows from these two cases that the minimum value must occur at one of the points  $x_1, \dots, x_n$ , rather than at the endpoints  $A$  or  $B$ .

Finally, consider one of the intervals  $(x_r, x_{r+1})$  where  $r$  is 1 or 2 or  $\dots$  or  $n-1$ . For  $x$  in this interval,  $x - x_i$  is positive for those  $i$  with  $i \leq r$  and negative for those  $i$  with  $i > r$ , so that here,

$$D(x) = \{N_1(x - x_1) + \dots + N_r(x - x_r)\} - \{N_{r+1}(x - x_{r+1}) + \dots + N_n(x - x_n)\}$$

which is a straight line with slope

$$S_r = (N_1 + \dots + N_r) - (N_{r+1} + \dots + N_n).$$

Looking at the slopes we see that

$$\begin{aligned} S_0 &= -(N_1 + \dots + N_n), \\ S_1 &= N_1 - (N_2 + \dots + N_n) = 2N_1 + S_0 > S_0, \\ S_2 &= N_1 + N_2 - (N_3 + \dots + N_n) = 2N_2 + S_1 > S_1, \end{aligned}$$

and in general  $S_{r+1} = 2N_r + S_r > S_r$ . This means that the slopes of the line segments which go to making up  $D(x)$  are strictly increasing so there are just two possibilities:

1. There is exactly one optimal location (this happens if no  $S_r = 0$ )
2. There are exactly two adjacent  $x$ 's, say  $x_i$  and  $x_{i+1}$ , where  $D(x)$  has the same value. In this case, any location in the closed interval  $[x_i, x_{i+1}]$  will give an optimal location.

By doing this further analysis we have got more understanding about what is going on in the problem. In addition, we have another way of finding the optimal location which uses the additional information. The optimal location occurs at  $x_i$  if  $S_{i-1} < 0$  and  $S_i > 0$ , (or else anywhere in  $[x_i, x_{i+1}]$  if  $S_i = 0$ ). How can we test for this?

1. Calculate  $N_1 + \dots + N_n$  and set  $T_0 = S_0/2 = -(N_1 + \dots + N_n)/2$ ; (this takes  $n+1$  arithmetic operations).

2. Repeatedly add  $N_i$ 's to get  $T_1, T_2, \dots$  until  $T_i \geq 0$ ; this is at most  $(n-1)$  further operations. (We know  $T_n = S_n/2$  is positive so the change must occur among  $T_1, \dots, T_{n-1}$ .)

Therefore, using this method, we can find the optimal location in at most  $2n$  operations. Evaluating  $D(x)$  directly at each of  $x_1, \dots, x_n$  and then finding the smallest value requires approximately  $2n^2$  operations. If  $n = 10$ , this gives approximately 200 operations as opposed to 20 by the other method. When  $n = 100$ , direct calculation requires approximately 20,000 as opposed to 200 !

The second solution is clearly more efficient. Furthermore, we now understand much more. For example, our decision about which station is the optimal one doesn't depend directly on the locations  $x_i$ ; it only depends on the values  $N_i$  and on the order in which they occur! We wouldn't have guessed that up front. The mathematical analysis has given us *understanding* about how the system works and not just numbers.