1. Let (X, d) be a metric space, and $R > 0$. Prove that the collection

$$\mathcal{B}_R = \left\{B_d(x, r) : \ x \in X,\ 0 < r < R\right\}$$

is a basis for the topology on X induced by d.

2. Let $n \in \mathbb{N}$, and assume that (X_i, d_i) is a metric space for $i = 1$, ..., n. Prove that

$$d(x, y) = \max\{d_1(x_1, y_1), \cdots, d_n(x_n, y_n)\}$$

defines a metric on $\prod_{i=1}^{n} X_i$.

3. Assume that for each $i \in \mathbb{N}$, (X_i, d_i) is a metric space. Prove that

$$d(x, y) = \sum_{i=1}^{+\infty} \frac{\overline{d_i}(x_i, y_i)}{2^i}$$

defines a metric on $\prod_{i=1}^{+\infty} X_i$.