Geometric Aspects of Sturm-Liouville Problems

V. Natural Loops of Boundary Conditions for Monotonicity of Eigenvalues and Their Applications

Wujian Peng, Mihai Racovitan and Hongyou Wu

Dedicated to Professor Zhi-Jiang Cao on the occasion of his seventieth birthday

Abstract. For a given Sturm-Liouville equation, we prove several new monotonicity results about its eigenvalues on the space of self-adjoint boundary conditions, and show that the monotonicity does not change along certain natural loops in this space. These results yield a general way for finding and/or proving inequalities among eigenvalues for different boundary conditions. For example, we obtain a natural explanation and a short proof of the inequalities among eigenvalues for coupled boundary conditions and those for separated boundary conditions established by Eastham, Kong, Wu and Zettl in [6]; moreover, several new sequences of inequalities among eigenvalues are derived.

This paper is a continuation of [11] and [5]. We study self-adjoint Sturm-Liouville problems (SLP's) associated with Sturm-Liouville equations (SLE's) of the form

$$-(fy')' + qy = \lambda w y \text{ on } (a,b), \tag{0.1}$$

where

$$-\infty \le a < b \le \infty, \qquad 1/f,\ q,\ w \in \mathrm{L}((a,b),\mathbb{R}), \qquad f, w > 0 \text{ a.e. on } (a,b), \tag{0.2}$$

and $\lambda \in \mathbb{C}$ is the so-called spectral parameter. Here, for an interval $J \subseteq \mathbb{R}$, we denote by $\mathrm{L}(J,\mathbb{R})$ the space of Lebesgue integrable real functions on J.

The following inequalities are well-known:

$$\begin{aligned} \lambda_1^{\boldsymbol{N}} \le \lambda_1^{\boldsymbol{P}} < \lambda_1^{\mathrm{SP}} \le \boldsymbol{\{}\lambda_1^{\boldsymbol{D}}, \lambda_2^{\boldsymbol{N}}\boldsymbol{\}} \le \lambda_2^{\mathrm{SP}} < \lambda_2^{\boldsymbol{P}} \le \boldsymbol{\{}\lambda_2^{\boldsymbol{D}}, \lambda_3^{\boldsymbol{N}}\boldsymbol{\}} \\ \le \lambda_3^{\boldsymbol{P}} < \lambda_3^{\mathrm{SP}} \le \boldsymbol{\{}\lambda_3^{\boldsymbol{D}}, \lambda_4^{\boldsymbol{N}}\boldsymbol{\}} \le \lambda_4^{\mathrm{SP}} < \lambda_4^{\boldsymbol{P}} \le \boldsymbol{\{}\lambda_4^{\boldsymbol{D}}, \lambda_5^{\boldsymbol{N}}\boldsymbol{\}} \le \cdots, \end{aligned} \tag{0.3}$$

where $\{\lambda_n^{\boldsymbol{P}}\}_{n\in\mathbb{N}}$, $\{\lambda_n^{\mathrm{SP}}\}_{n\in\mathbb{N}}$, $\{\lambda_n^{\boldsymbol{D}}\}_{n\in\mathbb{N}}$ and $\{\lambda_n^{\boldsymbol{N}}\}_{n\in\mathbb{N}}$ are the eigenvalues for the periodic, semi-periodic, Dirichlet and Neumann boundary conditions (BC's), respectively. Here, for any list L of numbers, the notation $\boldsymbol{\{}L\boldsymbol{\}}$ with bold faced braces means each of the numbers in L; for example, for any two numbers c_1 and c_2, $\{c_1, c_2\}$ means each of c_1 and c_2. The above inequalities have been extended to the case of an arbitrary coupled self-adjoint BC in [6] (see also [7]). A key point in the work [6] is the identification of two separated self-adjoint BC's corresponding to the given coupled self-adjoint BC that play, in these general inequalities, the role of the Dirichlet and Neumann BC's in the above classical inequalities.

Such general inequalities have also been found for singular SLP's with regular or limit circle non-oscillatory end-points [10], for left-definite SLP's with an indefinite weight function w [12], for self-adjoint SLP's whose leading coefficient function f changes sign [4], for singular left-definite SLP's with an indefinite w [14], and for generalized SLP's with finite spectra [13].

In addition to being interesting by their own, this type of general inequalities have two applications. First, they yield an algorithm for computing the eigenvalue with a given index (say, the eighth eigenvalue) for coupled self-adjoint BC's. See [2] for an implementation of this algorithm in some cases. For separated self-adjoint BC's, such an algorithm is provided by the Prüfer angle characterization of eigenvalues. Second, using such inequalities one can generalize asymptotic formulas for eigenvalues from the case of separated self-adjoint BC's to the case of coupled self-adjoint BC's. For example, in the situation where f changes sign, such a generalization of the asymptotic formulas from [1] and [3] is achieved for the first time in [4].

However, in the papers [6], [7], [10], [12], [4], [14] and [13], no explanation is given as to why the separated self-adjoint BC's used in the inequalities should be chosen in the way first given in [6]. An explanation is of crucial importance if one wants to generalize these inequalities to the case where the differential equation (DE) in the spectral problem is of a higher order.

In this paper, using the geometric structure on the space $\mathcal{B}^{\mathbb{C}}$ of self-adjoint BC's given in [11], we first obtain some natural loops in $\mathcal{B}^{\mathbb{C}}$. The intersections of some of these loops with the space of separated self-adjoint BC's are exactly the two separated self-adjoint BC's used in the general inequalities established in [6]. This gives a natural explanation of these two BC's. Then, starting from derivative formulas for eigenvalues given in [11], we prove several monotonicity results about the eigenvalues of (0.1) on the above natural loops, see Theorem 3.38. These results, together with the complete characterization of the discontinuities of the indexed eigenvalues given in [9], yield a general way for finding and/or proving inequalities among eigenvalues for different BC's. In particular, we obtain a short proof of the general inequalities in [6]. Moreover, new general inequalities (parallel to the general inequalities in [6]), and new inequalities among the eigenvalues for a generic coupled self-adjoint BC and those for two corresponding special coupled self-adjoint BC's (i.e., two on the so-called jump set) are derived in this way, see Theorems 4.53 and 4.63.

One can give a similar proof of the general inequalities found in [4], and here we omit the details. Note that the general inequalities in [10], [12] and [14] are derived from the ones in [6]. Moreover, such inequalities can also be established for SLP's with limit circle oscillatory end-points, see [16].

A natural remaining question about the general inequalities in [6] is: when does an equality in them hold? Combining techniques in [11] and this paper, we can give a complete answer to this question, see [17].

This paper's monotonicity results about eigenvalues on natural loops of self-adjoint BC's and general way for finding and/or proving inequalities among eigenvalues for different self-adjoint BC's can be generalized to the case where the DE in the spectral problem is of a higher order. We will pursue this in a further publication.

The organization of this paper is as follows. In Section 1, we introduce our notation and recall some basic results. To illustrate the main ideas of our approach, in Section 2 some inequalities among eigenvalues for separated self-adjoint BC's are established using known monotonicity results. In Section 3, we present some natural loops in the space of self-adjoint BC's and prove new monotonicity results about eigenvalues, while Section 4 is devoted to a short proof of the general inequalities in [6] and the derivation of some new inequalities.

§1. Notation and Basic Results

For any $m, n \in \mathbb{N}$, we use $\mathrm{M}_{m,n}(\mathbb{C})$ to denote the vector space of m by n complex matrices, and $\mathrm{M}^*_{m,n}(\mathbb{C})$ its open subset consisting of the elements with the maximum rank $\min\{m, n\}$; while $\mathrm{M}_{m,n}(\mathbb{R})$ and $\mathrm{M}^*_{m,n}(\mathbb{R})$ are the real analogs of $\mathrm{M}_{m,n}(\mathbb{C})$ and $\mathrm{M}^*_{m,n}(\mathbb{C})$, respectively. When a capital Greek or Latin letter other than Y stands for a matrix, the entries of the matrix will be denoted by the corresponding lower case letter with two indices. Let $\mathrm{GL}(2, \mathbb{C})$ be the set of invertible complex matrices in dimension 2, and $\mathrm{SL}(2, \mathbb{R})$ its subset consisting of the real elements having determinant 1. For a complex matrix A, A^* stands for its complex conjugate transpose.

By a solution of (0.1) we mean a function y on (a, b) such that y and fy' are absolutely continuous on all compact subintervals of (a, b) and satisfy (0.1) a.e.. The regularity conditions in (0.2) imply that every solution y and its quasi-derivative fy' have finite limits at the both end-points a and b, and any initial-value problem for (0.1) on $[a, b]$ has a unique solution.

For each $\lambda \in \mathbb{C}$, let $\phi_{11}(\cdot, \lambda)$ and $\phi_{12}(\cdot, \lambda)$ be the solutions of (0.1) determined by the initial conditions

$$\phi_{11}(a, \lambda) = 1,\ (f\phi'_{11})(a, \lambda) = 0, \qquad \phi_{12}(a, \lambda) = 0,\ (f\phi'_{12})(a, \lambda) = 1. \tag{1.1}$$

We denote $f\phi'_{11}$ by ϕ_{21} and $f\phi'_{12}$ by ϕ_{22}. Set

$$\Phi(t, \lambda) = \begin{pmatrix} \phi_{11}(t, \lambda) & \phi_{12}(t, \lambda) \\ \phi_{21}(t, \lambda) & \phi_{22}(t, \lambda) \end{pmatrix}, \quad t \in [a, b],\ \lambda \in \mathbb{C}. \tag{1.2}$$

For each $t \in [a, b]$, $\Phi(t, \lambda)$ is an entire matrix function of λ. Moreover, $\Phi(t, \lambda) \in \mathrm{SL}(2, \mathbb{R})$ for $t \in [a, b]$ and $\lambda \in \mathbb{R}$.

For a solution y of (0.1), we set

$$Y = \begin{pmatrix} y \\ fy' \end{pmatrix}. \tag{1.3}$$

The self-adjoint BC's are represented by linear algebraic systems of the form

$$A\,Y(a) + B\,Y(b) = 0, \tag{1.4}$$

where $(A\,|\,B) \in \mathrm{M}^*_{2,4}(\mathbb{C})$ satisfies

$$A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B^*. \tag{1.5}$$

Following [11], we take the quotient space

$$\mathrm{GL}(2,\mathbb{C}) \Big\backslash \mathrm{M}^*_{2,4}(\mathbb{C}) = \Big\{ \{(TA\,|\,TB);\, T \in \mathrm{GL}(2,\mathbb{C})\};\ (A\,|\,B) \in \mathrm{M}^*_{2,4}(\mathbb{C}) \Big\} \tag{1.6}$$

as the space of BC's, i.e., *each BC is an equivalence class of coefficient matrices of linear algebraic systems of the form* (1.4) *with* $(A\,|\,B) \in \mathrm{M}^*_{2,4}(\mathbb{C})$. The BC represented by (1.4) will be denoted by $[A\,|\,B]$. Note here that square brackets, not parentheses, are used. Usual bold faced capital Latin letters, such as $\boldsymbol{A}$, will also be used for BC's. The space $\mathcal{B}^{\mathbb{R}}$ of real self-adjoint BC's consists of the separated real BC's and the coupled BC's of the form $[K\,|\,-I]$ with $K \in \mathrm{SL}(2,\mathbb{R})$. The space $\mathcal{B}^{\mathbb{C}}$ of complex self-adjoint BC's is made of the real self-adjoint BC's and the non-real BC's of the form $[\mathrm{e}^{\mathrm{i}\gamma}K\,|\,-I]$ with $\gamma \in (0,\pi)$ and $K \in \mathrm{SL}(2,\mathbb{R})$. By [11], $\mathcal{B}^{\mathbb{C}}$ is a compact real analytic manifold and can be obtained by "gluing" its open sets

$$\mathcal{O}^{\mathbb{C}}_1 = \mathcal{O}^{\mathbb{C}}_6 = \left\{ [\mathrm{e}^{\mathrm{i}\gamma}K\,|\,-I];\ \gamma \in [0,\pi),\, K \in \mathrm{SL}(2,\mathbb{R}) \right\}, \tag{1.7}$$

$$\mathcal{O}^{\mathbb{C}}_2 = \left\{ \begin{bmatrix} 1 & a_{12} & 0 & \overline{z} \\ 0 & z & -1 & b_{22} \end{bmatrix};\ a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.8}$$

$$\mathcal{O}^{\mathbb{C}}_3 = \left\{ \begin{bmatrix} 1 & a_{12} & -\overline{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix};\ a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\}, \tag{1.9}$$

$$\mathcal{O}^{\mathbb{C}}_4 = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & -\overline{z} \\ z & 0 & -1 & b_{22} \end{bmatrix};\ a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.10}$$

$$\mathcal{O}^{\mathbb{C}}_5 = \left\{ \begin{bmatrix} a_{11} & 1 & \overline{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix};\ a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\} \tag{1.11}$$

via the coordinate transformations among these open sets. Note that the topology on the open set in (1.7) is the one induced from the usual topology on $\mathrm{M}_{2,2}(\mathbb{C})$, and each of the four open sets in (1.8)–(1.11) can be identified with $\mathbb{R}^4$. Open sets $\mathcal{O}^{\mathbb{R}}_i$, $i = 1, ..., 6$, of $\mathcal{B}^{\mathbb{R}}$ can be defined using (1.7)–(1.11) with $\omega = 0$ and $\mathbb{C}$ replaced by $\mathbb{R}$. Then, $\mathcal{B}^{\mathbb{R}}$ is a compact real analytic manifold and can be obtained by gluing these open sets via the coordinate transformations among them, and each of $\mathcal{O}^{\mathbb{R}}_2$, ..., $\mathcal{O}^{\mathbb{R}}_5$ can be identified with $\mathbb{R}^3$.

In this paper, we always consider the arbitrarily fixed SLE (0.1) satisfying (0.2), and hence call the eigenvalues of the SLP consisting of (0.1) and a BC the eigenvalues for the BC. The following result is well-known and can be verified directly.

Theorem 1.12. *A number $\lambda \in \mathbb{C}$ is an eigenvalue for (1.4) if and only if*

$$\varDelta(\lambda) := \det(A + B\varPhi(b, \lambda)) = 0. \tag{1.13}$$

For its importance, we will call the entire function $\varDelta$, unique up to a non-zero constant multiple, the **characteristic function** for the BC $[A \,|\, B]$. Recall that the **analytic multiplicity** (or just **multiplicity**) of an isolated eigenvalue is the order of the eigenvalue as a zero of $\varDelta$. An eigenvalue is said to be **simple** if it has multiplicity 1, while the eigenvalues of multiplicity 2 are called **double** eigenvalues. When we count the (isolated) eigenvalues for a BC in a domain in $\mathbb{C}$, their multiplicities are taken into account. The linear space spanned by the eigenfunctions for an eigenvalue is called the **eigenspace** for the eigenvalue. The **geometric multiplicity** of an eigenvalue is defined to be the dimension of its eigenspace, which is either 1 or 2. The linearity of the DE (0.1) implies that the geometric multiplicity of any eigenvalue for a separated BC is 1.

For a coupled self-adjoint BC $[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I]$, where $K \in \mathrm{SL}(2, \mathbb{R})$ and $\gamma \in \mathbb{R}$, if we rewrite it as $[\mathrm{e}^{\mathrm{i}\gamma/2}K \,|\, -\mathrm{e}^{-\mathrm{i}\gamma/2}I]$, then straightforward calculations yield that the characteristic function for it is

$$\varDelta(\lambda) = 2\cos\gamma - k_{22}\phi_{11}(b, \lambda) + k_{21}\phi_{12}(b, \lambda) + k_{12}\phi_{21}(b, \lambda) - k_{11}\phi_{22}(b, \lambda). \tag{1.14}$$

Thus, the eigenvalues for $[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I]$ are the same as those for $[\mathrm{e}^{-\mathrm{i}\gamma}K \,|\, -I]$, counting multiplicity.

The following result is also well-known.

Theorem 1.15. *For each $\boldsymbol{A} \in \mathcal{B}^{\mathbb{C}}$, there are infinitely many eigenvalues for $\boldsymbol{A}$, and they are all real and can be ordered to form a non-decreasing sequence*

$$\lambda_1(\boldsymbol{A}) \le \lambda_2(\boldsymbol{A}) \le \lambda_3(\boldsymbol{A}) \le \cdots \tag{1.16}$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its analytic multiplicity.

The following result is from Theorem 5.5 in [11].

Theorem 1.17. *The analytic multiplicity of each eigenvalue for any self-adjoint boundary condition is equal to the geometric multiplicity of the eigenvalue.*

So, for any eigenvalue for a self-adjoint BC, we will not distinguish its multiplicity and geometric multiplicity.

The next result is a slight generalization of a special case of Theorem 3.1 in [15] or Theorem 3.2 in [8], and can be proved using Rouché's Theorem from complex analysis.

Theorem 1.18. *Let $\mathcal{R} \subset \mathbb{R}$ be a bounded open set such that its boundary does not contain any eigenvalue for a given self-adjoint boundary condition $\boldsymbol{A}$, and $n \geq 0$ the number of eigenvalues for $\boldsymbol{A}$ in $\mathcal{R}$. Then there exists a neighborhood $\mathcal{N}$ of $\boldsymbol{A}$ in $\mathcal{B}^{\mathbb{C}}$ such that any boundary condition in $\mathcal{N}$ also has exactly n eigenvalues in $\mathcal{R}$.*

Remark 1.19. Let λ_* be an eigenvalue for a self-adjoint BC $\boldsymbol{A}_0$ and n its multiplicity. Pick a small $\epsilon > 0$ such that $\boldsymbol{A}_0$ has exactly n eigenvalues in the interval $[\lambda_* - \epsilon, \lambda_* + \epsilon]$. Then, by Theorem 1.18, there is a connected neighborhood $\mathcal{O}$ of $\boldsymbol{A}_0$ in $\mathcal{B}^{\mathbb{C}}$ such that each BC in $\mathcal{O}$ has exactly n eigenvalues in $(\lambda_* - \epsilon, \lambda_* + \epsilon)$. Thus, there are continuous functions Λ_1, ..., $\Lambda_n : \mathcal{O} \to \mathbb{R}$ defined on $\mathcal{O}$ such that

1) $\Lambda_1(\boldsymbol{A}_0) = \cdots = \Lambda_n(\boldsymbol{A}_0) = \lambda_*$;
2) $\Lambda_1(\boldsymbol{A}) \leq \cdots \leq \Lambda_n(\boldsymbol{A})$ for any $\boldsymbol{A} \in \mathcal{O}$;
3) for each $\boldsymbol{A} \in \mathcal{O}$, $\Lambda_1(\boldsymbol{A})$, ... and $\Lambda_n(\boldsymbol{A})$ are eigenvalues for $\boldsymbol{A}$.

From Theorem 1.17 we see that $n = 1$ if $\boldsymbol{A}_0$ is a separated BC. In any case, locally, they are the only such functions and are called the **continuous eigenvalue branches** through λ_*.

We can apply the above ideas to any finite number of eigenvalues for a self-adjoint BC to get similar conclusions.

§2. Illustration of Main Ideas

In this section, as an illustration of the main ideas of our approach, from well-known monotonicity results we deduce some inequalities among eigenvalues for separated self-adjoint BC's.

Each separated self-adjoint BC can be written in the form

$$\boldsymbol{S}_{\alpha,\beta} := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \end{bmatrix} \tag{2.1}$$

with $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$. For example, the Dirichlet BC $\boldsymbol{D} = \boldsymbol{S}_{0,\pi}$. Note that the space

$$\mathcal{T} = \left\{\boldsymbol{S}_{\alpha,\beta};\ \alpha \in [0, \pi),\ \beta \in (0, \pi]\right\} \tag{2.2}$$

of separated self-adjoint BC's is diffeomorphic to the torus.

The following results can be shown using the Prüfer angle characterization of the eigenvalues for separated self-adjoint BC's and Theorem 1.17.

Lemma 2.3. *Let $n \in \mathbb{N}$. As a function of (α, β), $\lambda_n(\boldsymbol{S}_{\alpha,\beta})$ is continuous on $[0, \pi) \times (0, \pi]$, strictly decreasing in α, and strictly increasing in β. Moreover, for each $\alpha \in [0, \pi)$,*

$$\lim_{\beta \to 0^+} \lambda_1(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\beta \to 0^+} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{\alpha,\pi}) \text{ if } n \geq 2, \tag{2.4}$$

and for each $\beta \in (0, \pi]$,

$$\lim_{\alpha \to \pi^-} \lambda_1(\boldsymbol{S}_{\alpha,\beta}) = -\infty, \qquad \lim_{\alpha \to \pi^-} \lambda_n(\boldsymbol{S}_{\alpha,\beta}) = \lambda_{n-1}(\boldsymbol{S}_{0,\beta}) \text{ if } n \geq 2. \tag{2.5}$$

In the rest of this section, we will abbreviate $\lambda_n(\boldsymbol{S}_{\alpha,\beta})$ as $\lambda_n(\alpha, \beta)$ for any $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$.

Fix an $n \in \mathbb{N}$ and an $\alpha \in [0, \pi)$. By Lemma 2.3, $\lambda_n(\alpha, \beta)$ is a strictly increasing function of β on $(0, \pi]$. Therefore, for each $\beta \in (0, \pi]$,

$$\lim_{\tau \to 0^+} \lambda_n(\alpha, \tau) < \lambda_n(\alpha, \beta) \leq \lambda_n(\alpha, \pi). \tag{2.6}$$

Hence, from (2.4) we obtain that

$$-\infty < \lambda_1(\alpha, \beta) \leq \lambda_1(\alpha, \pi), \qquad \lambda_{n-1}(\alpha, \pi) < \lambda_n(\alpha, \beta) \leq \lambda_n(\alpha, \pi) \text{ if } n \geq 2, \tag{2.7}$$

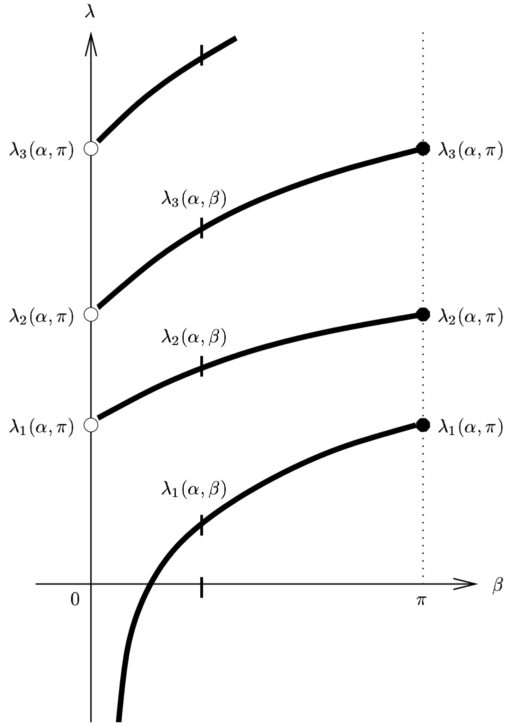

Figure 2.8. Eigenvalues as Functions of β, Case 1

or equivalently,

$$-\infty < \lambda_1(\alpha, \beta) \leq \lambda_1(\alpha, \pi) < \lambda_2(\alpha, \beta) \leq \lambda_2(\alpha, \pi) < \cdots . \tag{2.9}$$

These inequalities are interlacing relations among the eigenvalues for an arbitrary BC $\boldsymbol{S}_{\alpha,\beta}$ in $\mathcal{T}$ and those for the corresponding BC $\boldsymbol{S}_{\alpha,\pi}$, which is Dirichlet at b.

Similarly, for any $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$,

$$-\infty < \lambda_1(\alpha, \beta) \leq \lambda_1(0, \beta) < \lambda_2(\alpha, \beta) \leq \lambda_2(0, \beta) < \cdots . \tag{2.10}$$

These inequalities are interlacing relations among the eigenvalues for an arbitrary BC $\boldsymbol{S}_{\alpha,\beta}$ in $\mathcal{T}$ and those for the corresponding BC $\boldsymbol{S}_{0,\beta}$, which is Dirichlet at a.

Therefore, for any $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$, combining (2.9) and (2.10) we obtain that

$$-\infty < \lambda_1(\alpha,\beta) \le \boldsymbol{\{}\lambda_1(\alpha,\pi), \lambda_1(0,\beta)\boldsymbol{\}} < \lambda_2(\alpha,\beta) \le \boldsymbol{\{}\lambda_2(\alpha,\pi), \lambda_2(0,\beta)\boldsymbol{\}} < \cdots . \tag{2.11}$$

It seems to us that these are the simplest inequalities of the same type as (0.3).

The eigenvalues for an arbitrary BC $\boldsymbol{S}_{\alpha,\beta}$ in $\mathcal{T}$ can be compared with those for other classes of BC's in $\mathcal{T}$. For example, in stead of using $\boldsymbol{S}_{\alpha,\pi}$ and $\boldsymbol{S}_{0,\beta}$, we can take BC's in $\mathcal{T}$ which are Neumann at an endpoint. We now present the results so obtained.

Fix an $n \in \mathbb{N}$ and an $\alpha \in [0,\pi)$. If $0 < \beta \le \frac{\pi}{2}$, then

$$\lambda_n(\alpha,\beta) \le \lambda_n(\alpha, \tfrac{\pi}{2}) < \lambda_n(\alpha,\pi) = \lim_{\tau \to 0^+} \lambda_{n+1}(\alpha,\tau) < \lambda_{n+1}(\alpha,\beta). \tag{2.12}$$

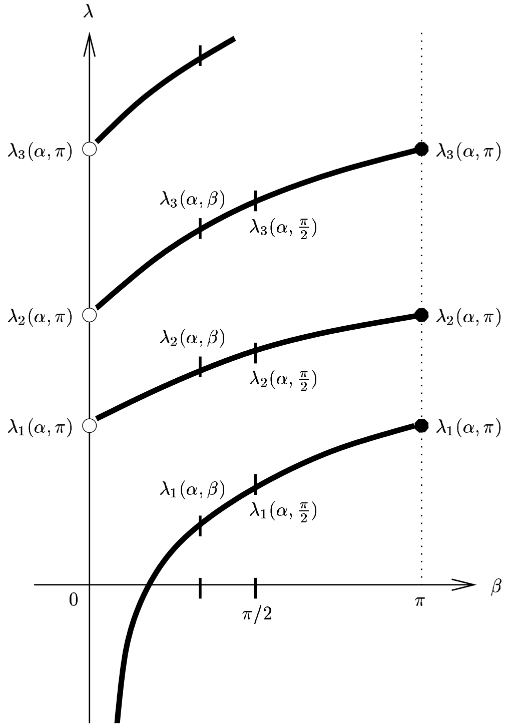

Figure 2.13. Eigenvalues as Functions of β, Case 2

Thus,

$$\lambda_1(\alpha,\beta) \le \lambda_1(\alpha, \tfrac{\pi}{2}) < \lambda_2(\alpha,\beta) \le \lambda_2(\alpha, \tfrac{\pi}{2}) < \cdots . \tag{2.14}$$

If $\frac{\pi}{2} < \beta \le \pi$, then

$$\lambda_n(\alpha, \tfrac{\pi}{2}) < \lambda_n(\alpha,\beta) \le \lambda_n(\alpha,\pi) = \lim_{\tau \to 0^+} \lambda_{n+1}(\alpha,\tau) < \lambda_{n+1}(\alpha, \tfrac{\pi}{2}). \tag{2.15}$$

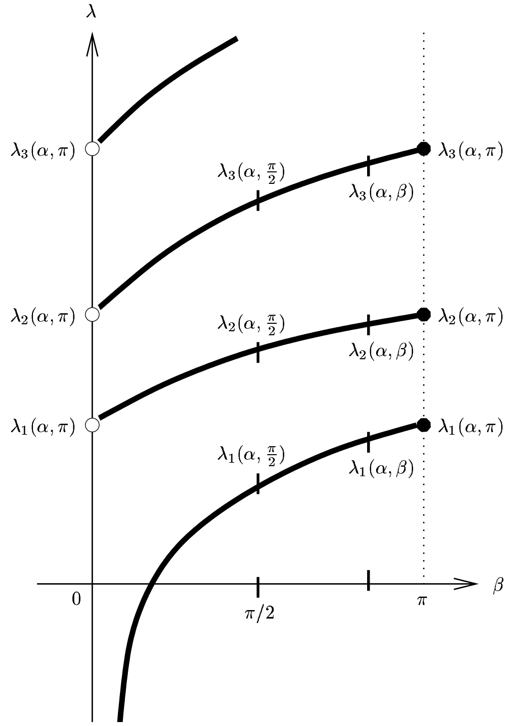

Figure 2.16. Eigenvalues as Functions of β, Case 3

So,

$$\lambda_1(\alpha, \tfrac{\pi}{2}) < \lambda_1(\alpha,\beta) < \lambda_2(\alpha, \tfrac{\pi}{2}) < \lambda_2(\alpha,\beta) < \cdots . \tag{2.17}$$

Similarly, fix a $\beta \in (0,\pi]$. If $0 \le \alpha < \frac{\pi}{2}$, then

$$\lambda_1(\tfrac{\pi}{2},\beta) < \lambda_1(\alpha,\beta) < \lambda_2(\tfrac{\pi}{2},\beta) < \lambda_2(\alpha,\beta) < \cdots ; \tag{2.18}$$

if $\frac{\pi}{2} \le \alpha < \pi$, then

$$\lambda_1(\alpha,\beta) \le \lambda_1(\tfrac{\pi}{2},\beta) < \lambda_2(\alpha,\beta) \le \lambda_2(\tfrac{\pi}{2},\beta) < \cdots . \tag{2.19}$$

To combine the inequalities in (2.14), (2.17), (2.18) and (2.19), there are four cases to discuss, according to the values of α and β. We omit the details.

Note that for each $\boldsymbol{S}_{\alpha,\beta}$,

$$\left\{\boldsymbol{S}_{\alpha,\tau};\ 0 < \tau \le \pi\right\} \ \text{ and } \ \left\{\boldsymbol{S}_{\tau,\beta};\ 0 \le \tau < \pi\right\} \tag{2.20}$$

are two natural loops in $\mathcal{T} \subset \mathcal{B}^{\mathbb{C}}$ passing through $\boldsymbol{S}_{\alpha,\beta}$. Hence, The above examples indicate a general way for finding and/or proving sequences of inequalities comparing the eigenvalues for BC's in one subset $\mathcal{S}_1$ of $\mathcal{B}^{\mathbb{C}}$ with those for BC's in another subset $\mathcal{S}_2$ of $\mathcal{B}^{\mathbb{C}}$, i.e., sequences of inequalities can be written down using graphs such as those in Figures 2.8, 2.13 and 2.16 whence the following three ingredients are obtained:

1) for each BC $\boldsymbol{A}$ in $\mathcal{S}_1$, there are (natural) loops in $\mathcal{B}^{\mathbb{C}}$ passing through $\boldsymbol{A}$ such that each of them contains exactly one BC from $\mathcal{S}_2$;
2) on each of these loops, all continuous eigenvalue branches have the same monotonicity, and each continuous eigenvalue branch does not change its monotonicity;
3) the complete characterization of the discontinuities of the indexed eigenvalues from [9] then tells us on such a loop, where and how the indices of the eigenvalues change.

Moreover, we remark that derivative formulas such as those given in [15], [8] and [11] are powerful tools for showing monotonicity results. To apply such formulas, the loops used in the above general method are required to be smooth a.e..

§3. Natural Loops of Boundary Conditions and Monotonicity of Eigenvalues

In this section, we obtain a natural explanation of the two separated self-adjoint BC's used in the general inequalities established in [6], and present new monotonicity results about continuous eigenvalue branches.

We begin by establishing some limits in the space $\mathcal{B}^{\mathbb{C}}$ of self-adjoint BC's.

Lemma 3.1. *In $\mathcal{B}^{\mathbb{C}}$, we have the following limits*:

$$\lim_{s\to\pm\infty}\begin{bmatrix}1 & s & 0 & \overline{z}\\ 0 & z & -1 & b_{22}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -1 & b_{22}\end{bmatrix} \qquad \forall\, z\in\mathbb{C}, b_{22}\in\mathbb{R}, \tag{3.2}$$

$$\lim_{s\to\pm\infty}\begin{bmatrix}1 & s & -\overline{z} & 0\\ 0 & z & b_{21} & -1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & b_{21} & -1\end{bmatrix} \qquad \forall\, z\in\mathbb{C}, b_{21}\in\mathbb{R}, \tag{3.3}$$

$$\lim_{s\to\pm\infty}\begin{bmatrix}s & 1 & 0 & -\overline{z}\\ z & 0 & -1 & b_{22}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & -1 & b_{22}\end{bmatrix} \qquad \forall\, z\in\mathbb{C}, b_{22}\in\mathbb{R}, \tag{3.4}$$

$$\lim_{s\to\pm\infty}\begin{bmatrix}s & 1 & \overline{z} & 0\\ z & 0 & b_{21} & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & b_{21} & -1\end{bmatrix} \qquad \forall\, z\in\mathbb{C}, b_{21}\in\mathbb{R}. \tag{3.5}$$

Proof. We verify (3.2) as follows:

$$\begin{aligned}\lim_{s\to\pm\infty}\begin{bmatrix}1 & s & 0 & \overline{z}\\ 0 & z & -1 & b_{22}\end{bmatrix} &= \lim_{s\to\pm\infty}\begin{bmatrix}1/s & 1 & 0 & \overline{z}/s\\ 0 & z & -1 & b_{22}\end{bmatrix}\\ &= \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & z & -1 & b_{22}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -1 & b_{22}\end{bmatrix}.\end{aligned} \tag{3.6}$$

Similarly, one can prove the other limits. ∎

Next, using the above limits, we give some natural loops in $\mathcal{B}^{\mathbb{C}}$.

Lemma 3.7. i) *Any boundary condition*

$$\begin{bmatrix}1 & a_{12} & 0 & \overline{z}\\ 0 & z & -1 & b_{22}\end{bmatrix} \in \mathcal{O}_2^{\mathbb{C}} \tag{3.8}$$

lies on a simple real-analytic loop in $\mathcal{B}^{\mathbb{C}}$*, i.e.,*

$$\mathcal{C}_{2,z,b_{22}} := \left\{ \begin{bmatrix} 1 & s & 0 & \overline{z} \\ 0 & z & -1 & b_{22} \end{bmatrix};\ s \in \mathbb{R} \right\} \bigcup \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix} \right\}. \tag{3.9}$$

ii) *Any boundary condition*

$$\begin{bmatrix} 1 & a_{12} & -\overline{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix} \in \mathcal{O}_3^{\mathbb{C}} \tag{3.10}$$

lies on a simple real-analytic loop in $\mathcal{B}^{\mathbb{C}}$*, i.e.,*

$$\mathcal{C}_{3,z,b_{21}} := \left\{ \begin{bmatrix} 1 & s & -\overline{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix};\ s \in \mathbb{R} \right\} \bigcup \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & b_{21} & -1 \end{bmatrix} \right\}. \tag{3.11}$$

iii) *Any boundary condition*

$$\begin{bmatrix} a_{11} & 1 & 0 & -\overline{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} \in \mathcal{O}_4^{\mathbb{C}} \tag{3.12}$$

lies on a simple real-analytic loop in $\mathcal{B}^{\mathbb{C}}$*, i.e.,*

$$\mathcal{C}_{4,z,b_{22}} := \left\{ \begin{bmatrix} s & 1 & 0 & -\overline{z} \\ z & 0 & -1 & b_{22} \end{bmatrix};\ s \in \mathbb{R} \right\} \bigcup \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix} \right\}. \tag{3.13}$$

iv) *Any boundary condition*

$$\begin{bmatrix} a_{11} & 1 & \overline{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix} \in \mathcal{O}_5^{\mathbb{C}} \tag{3.14}$$

lies on a simple real-analytic loop in $\mathcal{B}^{\mathbb{C}}$*, i.e.,*

$$\mathcal{C}_{5,z,b_{21}} := \left\{ \begin{bmatrix} s & 1 & \overline{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix};\ s \in \mathbb{R} \right\} \bigcup \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_{21} & -1 \end{bmatrix} \right\}. \tag{3.15}$$

PROOF. By (3.2), the definition (3.9) of $\mathcal{C}_{2,z,b_{22}}$ implies that $\mathcal{C}_{2,z,b_{22}}$ is a simple loop, and its part in $\mathcal{O}_2^{\mathbb{C}}$ is real-analytic. Thus, the only thing left is to verify the real-analyticity of the loop at its limit point, i.e., its point not in $\mathcal{O}_2^{\mathbb{C}}$. If $s \neq 0$, then

$$\begin{bmatrix} 1 & s & 0 & \overline{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} = \begin{bmatrix} 1/s & 1 & 0 & \overline{z}/s \\ -z/s & 0 & -1 & b_{22} - z\overline{z}/s \end{bmatrix}. \tag{3.16}$$

So, a neighborhood of the limit point in the loop is

$$\left\{ \begin{bmatrix} t & 1 & 0 & t\overline{z} \\ -tz & 0 & -1 & b_{22} - tz\overline{z} \end{bmatrix};\ t \in \mathbb{R} \right\}, \tag{3.17}$$

which implies the real-analyticity of the loop at the limit point.

Similarly, one can prove the other claims. ∎

Note that for $i = 2$, 3, 4 and 5, the loop $\mathcal{C}_{i,z,b_{22}}$ stays in $\mathcal{B}^{\mathbb{R}}$ if and only if $z \in \mathbb{R}$.

Now, we give the separated BC's on the above natural loops through a given coupled self-adjoint BC.

Lemma 3.18. *Let* $[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] \in \mathcal{B}^{\mathbb{C}}$, *where* $K \in \mathrm{SL}(2,\mathbb{R})$ *and* $\gamma \in \mathbb{R}$.

i) *If* $k_{12} \neq 0$, *then*

$$[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] = \begin{bmatrix} k_{11}/k_{12} & 1 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{12} & 0 \\ -\mathrm{e}^{\mathrm{i}\gamma}/k_{12} & 0 & k_{22}/k_{12} & -1 \end{bmatrix} \in \mathcal{O}_5^{\mathbb{C}}, \tag{3.19}$$

and the only separated boundary condition on the loop given in Lemma 3.7 iv) through this point is

$$\boldsymbol{S}_K := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_{22} & -k_{12} \end{bmatrix}. \tag{3.20}$$

ii) *If* $k_{22} \neq 0$, *then*

$$[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] = \begin{bmatrix} k_{21}/k_{22} & 1 & 0 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{22} \\ \mathrm{e}^{\mathrm{i}\gamma}/k_{22} & 0 & -1 & k_{12}/k_{22} \end{bmatrix} \in \mathcal{O}_4^{\mathbb{C}}, \tag{3.21}$$

and the only separated boundary condition on the loop given in Lemma 3.7 iii) through this point is also $\boldsymbol{S}_K$ *defined by* (3.20).

iii) *If* $k_{11} \neq 0$, *then*

$$[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] = \begin{bmatrix} 1 & k_{12}/k_{11} & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{11} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\gamma}/k_{11} & k_{21}/k_{11} & -1 \end{bmatrix} \in \mathcal{O}_3^{\mathbb{C}}, \tag{3.22}$$

and the only separated boundary condition on the loop given in Lemma 3.7 ii) through this point is

$$\boldsymbol{T}_K := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{11} \end{bmatrix}. \tag{3.23}$$

iv) *If* $k_{21} \neq 0$, *then*

$$[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] = \begin{bmatrix} 1 & k_{22}/k_{21} & 0 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{21} \\ 0 & -\mathrm{e}^{\mathrm{i}\gamma}/k_{21} & -1 & k_{11}/k_{21} \end{bmatrix} \in \mathcal{O}_2^{\mathbb{C}}, \tag{3.24}$$

and the only separated boundary condition on the loop given in Lemma 3.7 i) through this point is also $\boldsymbol{T}_K$ *defined by* (3.23).

Proof: i) Since $k_{12} \neq 0$, the condition $\det K = 1$ implies that

$$[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I] = \begin{bmatrix} k_{11}/k_{12} & 1 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{12} & 0 \\ \mathrm{e}^{\mathrm{i}\gamma}k_{21} & \mathrm{e}^{\mathrm{i}\gamma}k_{22} & 0 & -1 \end{bmatrix} = \begin{bmatrix} k_{11}/k_{12} & 1 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{12} & 0 \\ -\mathrm{e}^{\mathrm{i}\gamma}/k_{12} & 0 & k_{22}/k_{12} & -1 \end{bmatrix}. \tag{3.25}$$

By Lemma 3.7, the only separated BC on the loop in (3.15) through this point is $\boldsymbol{S}_K$.

ii) Since $k_{22} \neq 0$, the condition $\det K = 1$ yields that

$$[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I] = \begin{bmatrix} k_{21}/k_{22} & 1 & 0 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{22} \\ \mathrm{e}^{\mathrm{i}\gamma}k_{11} & \mathrm{e}^{\mathrm{i}\gamma}k_{12} & -1 & 0 \end{bmatrix} = \begin{bmatrix} k_{21}/k_{22} & 1 & 0 & -\mathrm{e}^{-\mathrm{i}\gamma}/k_{22} \\ \mathrm{e}^{\mathrm{i}\gamma}/k_{22} & 0 & -1 & k_{12}/k_{22} \end{bmatrix}. \tag{3.26}$$

By Lemma 3.7, the only separated BC on the loop in (3.13) through this point is also $\boldsymbol{S}_K$.

iii) and iv) The proofs are very similar and hence omitted. ■

Therefore, *$\mathcal{O}_2^{\mathbb{C}}$, $\mathcal{O}_3^{\mathbb{C}}$, $\mathcal{O}_4^{\mathbb{C}}$ and $\mathcal{O}_5^{\mathbb{C}}$ together form a natural atlas of coordinate systems on $\mathcal{B}^{\mathbb{C}}$ (similarly, a natural atlas of coordinate systems on $\mathcal{B}^{\mathbb{R}}$ consists of $\mathcal{O}_2^{\mathbb{R}}$, $\mathcal{O}_3^{\mathbb{R}}$, $\mathcal{O}_4^{\mathbb{R}}$ and $\mathcal{O}_5^{\mathbb{R}}$)*, and *the two separated BC's used in the general inequalities established in [6] are nothing but limits in coordinate directions of these coordinate systems (i.e., the intersections of natural loops in coordinate directions with the space of separated self-adjoint BC's).*

For a point p in a differential manifold M, we denote by T_pM the tangent space of M at p. From Remark 5.8 in [11] we obtain the following results.

Lemma 3.27. *If λ_* is an eigenvalue for $\boldsymbol{A} \in \mathcal{B}^{\mathbb{C}}$ of geometric multiplicity 1, then each continuous eigenvalue branch Λ through λ_* is differentiable at $\boldsymbol{A}$. Moreover, using a normalized eigenfunction u for λ_*, i.e., an eigenfunction for λ_* satisfying*

$$\int_a^b u(t)\overline{u}(t)w(t)\,\mathrm{d}t = 1, \tag{3.28}$$

we have the following derivative formulas: if $\boldsymbol{A} \in \mathcal{O}_2^{\mathbb{C}}$, then

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = \begin{pmatrix} \overline{u^{[1]}(a)} & \overline{u^{[1]}(b)} \end{pmatrix} \begin{pmatrix} h_{12} & \overline{h_{22}} \\ h_{22} & l_{22} \end{pmatrix} \begin{pmatrix} u^{[1]}(a) \\ u^{[1]}(b) \end{pmatrix} \tag{3.29}$$

for any $(H\,|\,L)$ in

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_2^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & h_{12} & 0 & \overline{h_{22}} \\ 0 & h_{22} & 0 & l_{22} \end{pmatrix};\ h_{12}, l_{22} \in \mathbb{R},\ h_{22} \in \mathbb{C} \right\}; \tag{3.30}$$

if $\boldsymbol{A} \in \mathcal{O}_3^{\mathbb{C}}$, then

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = \begin{pmatrix} \overline{u^{[1]}(a)} & \overline{u(b)} \end{pmatrix} \begin{pmatrix} h_{12} & -\overline{h_{22}} \\ -h_{22} & -l_{21} \end{pmatrix} \begin{pmatrix} u^{[1]}(a) \\ u(b) \end{pmatrix} \tag{3.31}$$

for any $(H\,|\,L)$ *in*

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_3^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & h_{12} & -\overline{h_{22}} & 0 \\ 0 & h_{22} & l_{21} & 0 \end{pmatrix};\ h_{12}, l_{21} \in \mathbb{R},\ h_{22} \in \mathbb{C} \right\}; \tag{3.32}$$

if $\boldsymbol{A} \in \mathcal{O}_4^{\mathbb{C}}$, *then*

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = \left(\overline{u(a)} \quad \overline{u^{[1]}(b)} \right) \begin{pmatrix} -h_{11} & \overline{h_{21}} \\ h_{21} & l_{22} \end{pmatrix} \begin{pmatrix} u(a) \\ u^{[1]}(b) \end{pmatrix} \tag{3.33}$$

for any $(H\,|\,L)$ *in*

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_4^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & 0 & -\overline{h_{21}} \\ h_{21} & 0 & 0 & l_{22} \end{pmatrix};\ h_{11}, l_{22} \in \mathbb{R},\ h_{21} \in \mathbb{C} \right\}; \tag{3.34}$$

if $\boldsymbol{A} \in \mathcal{O}_5^{\mathbb{C}}$, *then*

$$\mathrm{d}\Lambda\big|_{\boldsymbol{A}}((H\,|\,L)) = -\left(\overline{u(a)} \quad \overline{u(b)} \right) \begin{pmatrix} h_{11} & \overline{h_{21}} \\ h_{21} & l_{21} \end{pmatrix} \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} \tag{3.35}$$

for any $(H\,|\,L)$ *in*

$$\mathrm{T}_{\boldsymbol{A}}\mathcal{B}^{\mathbb{C}} = \mathrm{T}_{\boldsymbol{A}}\mathcal{O}_5^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & \overline{h_{21}} & 0 \\ h_{21} & 0 & l_{21} & 0 \end{pmatrix};\ h_{11}, l_{21} \in \mathbb{R},\ h_{21} \in \mathbb{C} \right\}. \tag{3.36}$$

The following result is from Theorem 4.1 in [11].

Lemma 3.37. *Let* $\lambda \in \mathbb{R}$. *Then, among all the complex boundary conditions,* $[\Phi(b,\lambda)\,|\,-I]$ *is the unique one that has* λ *as an eigenvalue of geometric multiplicity* 2.

As direct consequences of Lemmas 3.27 and 3.37, we have the following new monotonicity results.

Theorem 3.38. *If* $i = 2$ *or* 3, *then for any* $z \in \mathbb{C}$ *and* $c \in \mathbb{R}$, *each continuous eigenvalue branch on* $\mathcal{C}_{i,z,c}$ *is always increasing in the* a_{12}*-direction of* $\mathcal{O}_i^{\mathbb{C}}$; *if* $i = 4$ *or* 5, *then for any* $z \in \mathbb{C}$ *and* $c \in \mathbb{R}$, *each continuous eigenvalue branch on* $\mathcal{C}_{i,z,c}$ *is always decreasing in the* a_{11}*-direction of* $\mathcal{O}_i^{\mathbb{C}}$.

Proof. Let $z \in \mathbb{C}$ and $b_{22} \in \mathbb{R}$, and Λ be a continuous eigenvalue branch on (a simply connected part of) $\mathcal{C}_{2,z,b_{22}}$. Fix an $s_0 \in \mathbb{R}$ such that

$$\boldsymbol{A}(s) := \begin{bmatrix} 1 & s & 0 & \overline{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} \tag{3.39}$$

lies in the domain of Λ when $s = s_0$, and $\Lambda(\boldsymbol{A}(s_0))$ has geometric multiplicity 1. Assume that u is a normalized eigenfunction for $\Lambda(\boldsymbol{A}(s_0))$. Then, by (3.29),

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=s_0} \Lambda(\boldsymbol{A}(s)) = \overline{u^{[1]}(a)}\, u^{[1]}(a) \geq 0. \tag{3.40}$$

For s near s_0, $\Lambda(\boldsymbol{A}(s))$ has geometric multiplicity 1 and hence non-negative derivative. Thus, Λ is increasing near $\boldsymbol{A}(s_0)$. Note that $\mathcal{C}_{2,z,b_{22}}$ and the curve

$$\lambda \mapsto [\Phi(b,\lambda) \mid -I], \qquad \lambda \in \mathbb{R}, \tag{3.41}$$

are both real-analytic. So, either they agree completely, or their intersection is discrete in $\mathcal{C}_{2,z,b_{22}}$. In the latter case, Λ has geometric multiplicity 1 on a dense open subset of its domain, and hence Λ is always increasing. The former case can happen at most for one pair $z \in \mathbb{C}$ and $b_{22} \in \mathbb{R}$, and the always increasingness of Λ can be deduced from the latter case by perturbing z or b_{22}.

Similarly, we can show the other claims. ■

§4. Inequalities among Eigenvalues: Old and New

In this section, using the natural loops of BC's and monotonicity results on continuous eigenvalue branches from the last section, we give a new proof of the inequalities in [6] and derive some new inequalities.

In order to introduce the third ingredient needed, i.e., information about the discontinuities of λ_n, we let

$$\mathcal{F}^{\mathbb{C}}_{-} = \left\{ [\mathrm{e}^{\mathrm{i}\gamma}K \mid -I];\quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{11}k_{12} \leq 0,\ \gamma \in [0,\pi) \right\}, \tag{4.1}$$

$$\mathcal{H}^{\mathbb{C}}_{-} = \left\{ \begin{bmatrix} 1 & a_2 & -\overline{z} & 0 \\ 0 & z & b_1 & -1 \end{bmatrix};\quad a_2 \leq 0,\ b_1 \in \mathbb{R},\ z \in \mathbb{C} \right\}, \tag{4.2}$$

$$\mathcal{H}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{3} \setminus \mathcal{H}^{\mathbb{C}}_{-}, \tag{4.3}$$

$$\begin{aligned} \mathcal{J}^{\mathbb{C}} = {} & \left\{ [\mathrm{e}^{\mathrm{i}\gamma}K \mid -I];\quad K \in \mathrm{SL}(2,\mathbb{R}),\ k_{12} = 0,\ \gamma \in [0,\pi) \right\} \\ & \bigcup \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{B}^{\mathbb{R}};\quad a_2 b_2 = 0 \right\}. \end{aligned} \tag{4.4}$$

The following results are from Theorem 3.73 in [9].

Lemma 4.5. *The function λ_1 on $\mathcal{B}^{\mathbb{C}}$ is continuous on $\mathcal{B}^{\mathbb{C}} \setminus \mathcal{J}^{\mathbb{C}}$ and discontinuous at each point of $\mathcal{J}^{\mathbb{C}}$. For every $n \in \mathbb{N}$ satisfying $n \geq 2$, the function λ_n is continuous on $\mathcal{B}^{\mathbb{C}} \setminus \mathcal{J}^{\mathbb{C}}$ and at each coupled boundary condition in $\mathcal{J}^{\mathbb{C}}$ where $\lambda_n = \lambda_{n-1}$, and is discontinuous at any other point of $\mathcal{J}^{\mathbb{C}}$. More precisely, the restrictions of each λ_n to $\mathcal{F}^{\mathbb{C}}_{-}$ and $\mathcal{H}^{\mathbb{C}}_{-}$ are continuous, and for each $\boldsymbol{A} \in \mathcal{J}^{\mathbb{C}} \cap \mathcal{H}^{\mathbb{C}}_{-}$,*

$$\lim_{\mathcal{H}^{\mathbb{C}}_{+} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_1(\boldsymbol{B}) = -\infty, \qquad \lim_{\mathcal{H}^{\mathbb{C}}_{+} \ni \boldsymbol{B} \to \boldsymbol{A}} \lambda_n(\boldsymbol{B}) = \lambda_{n-1}(\boldsymbol{A}) \text{ if } n \geq 2. \tag{4.6}$$

Remark 4.7. The **jump set** $\mathcal{J}^{\mathbb{C}}$ does not depend on the SLE (0.1). It is easy to verify that $[A\,|\,B] \in \mathcal{B}^{\mathbb{C}}$ is in $\mathcal{J}^{\mathbb{C}}$ if and only if the (2,1)-entry in $B^{\mathrm{t}}A^{\mathrm{c}}$ is 0, where B^{t} is the transpose of B and A^{c} is the matrix of cofactors of A.

Note that the proof of the above results given in [9] does not use the inequalities in [6].

For $K \in \mathrm{SL}(2,\mathbb{R})$, let $\{\mu_n(K),\, n \in \mathbb{N}\}$ denote the eigenvalues for $\boldsymbol{S}_K$ given in (3.20) and $\{\nu_n(K),\, n \in \mathbb{N}\}$ the eigenvalues for $\boldsymbol{T}_K$ defined by (3.23). Note that $\mu_n(K) = \mu_n(-K)$ and $\nu_n(K) = \nu_n(-K)$ for all $n \in \mathbb{N}$. Moreover, from now on, $\lambda_n([\mathrm{e}^{\mathrm{i}\gamma}K\,|\,-I])$ will be abbreviated as $\lambda_n(\mathrm{e}^{\mathrm{i}\gamma}K)$. Now, we are ready to prove the essential parts of the inequalities in [6].

Theorem 4.8. *Let* $[\mathrm{e}^{\mathrm{i}\gamma}K\,|\,-I] \in \mathcal{B}^{\mathbb{C}}$, *where* $K \in \mathrm{SL}(2,\mathbb{R})$ *and* $\gamma \in \mathbb{R}$.

i) *We always have that*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \mu_1(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \mu_2(K) \le \cdots . \tag{4.9}$$

ii) *If* $k_{11}k_{12} \le 0$ *and* $k_{11} \ne 0$, *then*

$$\nu_1(K) \le \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \nu_2(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \cdots ; \tag{4.10}$$

and if $k_{11}k_{12} \ge 0$ *and* $k_{12} \ne 0$, *then*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \nu_1(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \nu_2(K) \le \cdots . \tag{4.11}$$

Note that if $K \in \mathrm{SL}(2,\mathbb{R})$, then k_{11} and k_{12} can not both equal 0, and hence either $k_{11}k_{12} \le 0$ and $k_{11} \ne 0$, or $k_{11}k_{12} \ge 0$ and $k_{12} \ne 0$. So, for each coupled self-adjoint BC, one and only one of (4.10) and (4.11) holds, unless $\lambda_n(\mathrm{e}^{\mathrm{i}\gamma}K) = \nu_n(K)$ for all $n \in \mathbb{N}$.

Proof. i) First, assume that $k_{12} \ne 0$. Let $a_{11} = k_{11}/k_{12}$, $z = -\mathrm{e}^{\mathrm{i}\gamma}/k_{12}$, $b_{21} = k_{22}/k_{12}$, and

$$\boldsymbol{A}(s) = \begin{bmatrix} s & 1 & \overline{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix} \qquad \forall\, s \in \mathbb{R}. \tag{4.12}$$

Then, by Lemmas 3.18 i) and 3.1, $[\mathrm{e}^{\mathrm{i}\gamma}K\,|\,-I] = \boldsymbol{A}(a_{11})$ is on the loop $\mathcal{C}_{5,z,b_{21}}$, and the only separated BC on this loop is $\boldsymbol{S}_K = \lim_{s\to\pm\infty}\boldsymbol{A}(s)$. Note that $\boldsymbol{S}_K$ is the only BC in $\mathcal{C}_{5,z,b_{21}} \setminus \mathcal{O}_5^{\mathbb{C}}$. By Remark 4.7 and direct calculations, $\mathcal{O}_5^{\mathbb{C}}$ does not intersect $\mathcal{J}^{\mathbb{C}}$, and hence for each $n \in \mathbb{N}$, $\lambda_n(\boldsymbol{A}(s))$ is continuous in s on $\mathbb{R}$. Thus, by Theorem 3.38, $\lambda_n(\boldsymbol{A}(s))$ is decreasing in s on $\mathbb{R}$. To see the limits of $\lambda_n(\boldsymbol{A}(s))$ at $\pm\infty$, we notice that for $s \ne 0$,

$$\boldsymbol{A}(s) = \begin{bmatrix} 1 & 1/s & \overline{z}/s & 0 \\ 0 & -z/s & b_{21} - z\overline{z}/s & -1 \end{bmatrix}, \tag{4.13}$$

and hence

$$\boldsymbol{A}(s) \in \mathcal{H}_-^{\mathbb{C}} \text{ if } s < 0, \qquad \boldsymbol{A}(s) \in \mathcal{H}_+^{\mathbb{C}} \text{ if } s > 0. \tag{4.14}$$

Since $\boldsymbol{S}_K \in \mathcal{J}^{\mathbb{C}} \cap \mathcal{H}_-^{\mathbb{C}}$ and $\lim_{s\to\pm\infty} \boldsymbol{A}(s) = \boldsymbol{S}_K$, (4.14) together with Lemma 4.5 yield that

$$\lim_{s\to-\infty} \lambda_n(\boldsymbol{A}(s)) = \lambda_n(\boldsymbol{S}_K) = \mu_n(K) \text{ for } n \in \mathbb{N}, \tag{4.15}$$

$$\lim_{s\to+\infty} \lambda_1(\boldsymbol{A}(s)) = -\infty, \tag{4.16}$$

$$\lim_{s\to+\infty} \lambda_n(\boldsymbol{A}(s)) = \lambda_{n-1}(\boldsymbol{S}_K) = \mu_{n-1}(K) \text{ for } n \geq 2. \tag{4.17}$$

Therefore, as in Section 2, we then deduce that for any $s \in \mathbb{R}$,

$$\lambda_1(\boldsymbol{A}(s)) \leq \mu_1(K) \leq \lambda_2(\boldsymbol{A}(s)) \leq \mu_2(K) \leq \cdots, \tag{4.18}$$

which implies (4.9).

If $k_{12} = 0$, then $k_{11}k_{22} = 1$, and hence

$$K_\epsilon := \begin{pmatrix} k_{11} & \epsilon \\ k_{21} & (1+\epsilon k_{21})/k_{11} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\ \forall\, \epsilon \neq 0, \qquad \lim_{\epsilon\to 0} K_\epsilon = K. \tag{4.19}$$

By the definition of $\boldsymbol{S}_K$, we see that

$$\boldsymbol{S}_{K_\epsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1-\epsilon k_{21} & \epsilon k_{11} \end{bmatrix}. \tag{4.20}$$

In the case where $k_{11} < 0$: as $\epsilon \to 0^+$,

$$[\mathrm{e}^{\mathrm{i}\gamma} K_\epsilon \,|\, -I] \longrightarrow [\mathrm{e}^{\mathrm{i}\gamma} K \,|\, -I] \text{ in } \mathcal{F}_-^{\mathbb{C}}, \qquad -1-\epsilon k_{21} \longrightarrow -1, \qquad \epsilon k_{11} \longrightarrow 0^-; \tag{4.21}$$

thus, by Lemmas 4.5 and 2.3, for each $n \in \mathbb{N}$,

$$\lambda_n(K_\epsilon) \longrightarrow \lambda_n(K), \qquad \mu_n(K_\epsilon) \longrightarrow \mu_n(K); \tag{4.22}$$

and hence we obtain (4.9) for this K from (4.9) for K_ϵ. In the case where $k_{11} > 0$: similarly, by letting $\epsilon \to 0^-$ we deduce (4.9) for this K from (4.9) for K_ϵ.

ii) First, assume that $k_{11} \neq 0$. Let $a_{12} = k_{12}/k_{11}$, $z = \mathrm{e}^{\mathrm{i}\gamma}/k_{11}$, $b_{21} = k_{21}/k_{11}$, and

$$\boldsymbol{A}(s) = \begin{bmatrix} 1 & s & -\overline{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix} \qquad \forall\, s \in \mathbb{R}. \tag{4.23}$$

Then, by Lemmas 3.18 iii) and 3.1, $[\mathrm{e}^{\mathrm{i}\gamma} K \,|\, -I] = \boldsymbol{A}(a_{12})$ is on the loop $\mathcal{C}_{3,z,b_{21}}$, and the only separated BC on this loop is $\boldsymbol{T}_K = \lim_{s\to\pm\infty} \boldsymbol{A}(s)$. By Remark 4.7 and direct calculations, $\mathcal{J}^{\mathbb{C}} \cap \mathcal{C}_{3,z,b_{21}} = \boldsymbol{A}(0)$. Hence, for each $n \in \mathbb{N}$, $\lambda_n(\boldsymbol{A}(s))$ is continuous in s on $(-\infty, 0)$ and $(0, +\infty)$, and

$$\lim_{s\to\pm\infty} \lambda_n(\boldsymbol{A}(s)) = \lambda_n(\boldsymbol{T}_K) = \nu_n(K). \tag{4.24}$$

Moreover, by Theorem 3.38, $\lambda_n(\boldsymbol{A}(s))$ is increasing in s on $(-\infty, 0)$ and $(0, +\infty)$. Since

$$\boldsymbol{A}(s) \in \mathcal{H}_-^{\mathbb{C}} \text{ if } s \leq 0, \qquad \boldsymbol{A}(s) \in \mathcal{H}_+^{\mathbb{C}} \text{ if } s > 0, \tag{4.25}$$

Lemma 4.5 yields that

$$\lim_{s\to 0^-} \lambda_n(\boldsymbol{A}(s)) = \lambda_n(\boldsymbol{A}(0)) \text{ for } n \in \mathbb{N}, \tag{4.26}$$

$$\lim_{s\to 0^+} \lambda_1(\boldsymbol{A}(s)) = -\infty, \qquad \lim_{s\to 0^+} \lambda_n(\boldsymbol{A}(s)) = \lambda_{n-1}(\boldsymbol{A}(0)) \text{ for } n \geq 2. \tag{4.27}$$

Thus, as in Section 2, we then deduce that for any $s \in \mathbb{R}$,

$$\nu_1(K) \leq \lambda_1(\boldsymbol{A}(s)) \leq \nu_2(K) \leq \lambda_2(\boldsymbol{A}(s)) \leq \cdots \text{ if } s \leq 0, \tag{4.28}$$

$$\lambda_1(\boldsymbol{A}(s)) \leq \nu_1(K) \leq \lambda_2(\boldsymbol{A}(s)) \leq \nu_2(K) \leq \cdots \text{ if } s > 0. \tag{4.29}$$

Note that (4.28) implies (4.10); while (4.29) yields (4.11) when $k_{11}k_{12} > 0$.

If $k_{11} = 0$, then $k_{12}k_{21} = -1$, and hence

$$K_\epsilon := \begin{pmatrix} \epsilon & k_{12} \\ (\epsilon k_{22} - 1)/k_{12} & k_{22} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \;\forall\, \epsilon \neq 0, \qquad \lim_{\epsilon\to 0} K_\epsilon = K. \tag{4.30}$$

By the definition of $\boldsymbol{T}_K$, we see that

$$\boldsymbol{T}_{K_\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon k_{22} - 1 & -\epsilon k_{12} \end{bmatrix}. \tag{4.31}$$

In the case where $k_{12} < 0$: as $\epsilon \to 0^-$,

$$[\mathrm{e}^{\mathrm{i}\gamma} K_\epsilon \,|\, -I] \longrightarrow [\mathrm{e}^{\mathrm{i}\gamma} K \,|\, -I] \notin \mathcal{J}^{\mathbb{C}}, \qquad \epsilon k_{22} - 1 \to -1, \qquad -\epsilon k_{12} \to 0^-; \tag{4.32}$$

thus, by Lemmas 4.5 and 2.3, for each $n \in \mathbb{N}$,

$$\lambda_n(K_\epsilon) \longrightarrow \lambda_n(K), \qquad \nu_n(K_\epsilon) \longrightarrow \nu_n(K); \tag{4.33}$$

and hence we obtain (4.11) for this K from (4.11) for K_ϵ. In the case where $k_{12} > 0$: similarly, by letting $\epsilon \to 0^+$ we deduce (4.11) for this K from (4.11) for K_ϵ. ∎

When ($k_{12} = 0$ and) $k_{22} \neq 0$, we can establish (4.9) using the natural loop $\mathcal{C}_{4,z,b_{22}}$; and when ($k_{11} = 0$ and) $k_{21} \neq 0$, one can prove (4.11) using the natural loop $\mathcal{C}_{2,z,b_{22}}$. However, the required arguments seem to be longer than the corresponding ones in the above proof.

Using (1.14), we can now prove the inequalities in [6], which are just refinements of (4.9)–(4.11).

THEOREM 4.34. *Let* $K \in \mathrm{SL}(2,\mathbb{R})$.

i) *If* $k_{11} > 0$ *and* $k_{12} \leq 0$, *then* $\lambda_1(K)$ *is simple, and for any* $\gamma \in (-\pi, 0) \cup (0, \pi)$, *we have that*

$$\begin{aligned} \nu_1(K) \leq \lambda_1(K) < \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_1(-K) &\leq \{\mu_1(K), \nu_2(K)\} \\ \leq \lambda_2(-K) < \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_2(K) &\leq \{\mu_2(K), \nu_3(K)\} \\ \leq \lambda_3(K) < \lambda_3(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_3(-K) &\leq \{\mu_3(K), \nu_4(K)\} \\ \leq \lambda_4(-K) < \lambda_4(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_4(K) &\leq \{\mu_4(K), \nu_5(K)\} \leq \cdots . \end{aligned} \tag{4.35}$$

ii) *If* $k_{11} \le 0$ *and* $k_{12} < 0$, *then* $\lambda_1(K)$ *is simple, and for any* $\gamma \in (-\pi, 0) \cup (0, \pi)$, *we have that*

$$\begin{aligned} &\lambda_1(K) < \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_1(-K) \le \{\mu_1(K), \nu_1(K)\} \le \\ &\lambda_2(-K) < \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_2(K) \le \{\mu_2(K), \nu_2(K)\} \le \\ &\lambda_3(K) < \lambda_3(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_3(-K) \le \{\mu_3(K), \nu_3(K)\} \le \\ &\lambda_4(-K) < \lambda_4(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_4(K) \le \{\mu_4(K), \nu_4(K)\} \le \cdots . \end{aligned} \tag{4.36}$$

iii) *If neither i) nor ii) applies to* K, *then either i) or ii) applies to* $-K$.

Proof. Note that each of (4.35) and (4.36) implies that $\lambda_1(K) \ne \lambda_2(K)$, and hence $\lambda_1(K)$ is simple in these cases.

Since K is fixed, we are going to abbreviate $\mu_n(K)$ as μ_n and $\nu_n(K)$ as ν_n. From (4.9)–(4.11) we obtain that if $k_{11}k_{12} \le 0$ and $k_{11} \ne 0$, then

$$\nu_1 \le \{\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K);\ \gamma \in \mathbb{R}\} \le \{\mu_1, \nu_2\} \le \{\lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K);\ \gamma \in \mathbb{R}\} \le \{\mu_2, \nu_3\} \le \cdots , \tag{4.37}$$

and if $k_{11}k_{12} \ge 0$ and $k_{12} \ne 0$, then

$$\{\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K);\ \gamma \in \mathbb{R}\} \le \{\mu_1, \nu_1\} \le \{\lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K);\ \gamma \in \mathbb{R}\} \le \{\mu_2, \nu_2\} \le \cdots . \tag{4.38}$$

By Lemma 3.1 in [9], if $k_{11} > 0$ and $k_{12} \le 0$, then

$$\lim_{\lambda \to -\infty} \Big(k_{22}\phi_{11}(b, \lambda) - k_{21}\phi_{12}(b, \lambda) - k_{12}\phi_{21}(b, \lambda) + k_{11}\phi_{22}(b, \lambda) \Big) = +\infty, \tag{4.39}$$

which together with (1.14) and (4.37) yield (4.35). We deduce ii) similarly, while iii) is evident since k_{11} and k_{12} can not both equal 0. ∎

So far, in our discussions, we have only used the loops along the a_{11}- and a_{12}-axes. By almost the same arguments, one can obtain similar inequalities in terms of the loops in the b_{21}- and b_{22}-directions. These similar inequalities can also be derived from the inequalities we already have by an easier method, which we now present.

After the substitution

$$t = -s \tag{4.40}$$

the SLP consisting of (0.1) and (1.4) is transformed to the SLP consisting of

$$-(\widetilde{p}\widetilde{y}')' + \widetilde{q}\widetilde{y} = \lambda \widetilde{w}\widetilde{y} \ \text{ on } (\widetilde{a}, \widetilde{b}), \tag{4.41}$$

$$BT \begin{pmatrix} \widetilde{y}(\widetilde{a}) \\ (\widetilde{p}\widetilde{y}')(\widetilde{a}) \end{pmatrix} + AT \begin{pmatrix} \widetilde{y}(\widetilde{b}) \\ (\widetilde{p}\widetilde{y}')(\widetilde{b}) \end{pmatrix} = 0, \tag{4.42}$$

where $\widetilde{a} = -b$, $\widetilde{b} = -a$,

$$\widetilde{y}(s) = y(-s),\ \widetilde{p}(s) = p(-s),\ \text{etc} \qquad \forall\, s \in (\widetilde{a}, \widetilde{b}), \tag{4.43}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.44}$$

So, the coupled BC $[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I]$ for (0.1) is transformed to the coupled BC

$$[-T \,|\, \mathrm{e}^{\mathrm{i}\gamma}KT] = [\mathrm{e}^{-\mathrm{i}\gamma}TK^{-1}T \,|\, -I] = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\gamma}k_{22} & \mathrm{e}^{-\mathrm{i}\gamma}k_{12} & -1 & 0 \\ \mathrm{e}^{-\mathrm{i}\gamma}k_{21} & \mathrm{e}^{-\mathrm{i}\gamma}k_{11} & 0 & -1 \end{bmatrix} \tag{4.45}$$

for (4.41), while transformed to the separated BC's

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_{11} & -k_{12} \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{22} \end{bmatrix} \tag{4.46}$$

for (4.41) associated with the coupled BC (4.45) for (4.41) are the separated BC's

$$\boldsymbol{U}_K := \begin{bmatrix} 0 & 0 & 1 & 0 \\ k_{11} & k_{12} & 0 & 0 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{4.47}$$

$$\boldsymbol{V}_K := \begin{bmatrix} 0 & 0 & 0 & -1 \\ -k_{21} & -k_{22} & 0 & 0 \end{bmatrix} = \begin{bmatrix} k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.48}$$

for (0.1), respectively. So, from Theorems 4.8 and 4.34 applied to the SLP consisting of (4.41) and (4.45) we immediately obtain the following results.

Theorem 4.49. *Let* $[\mathrm{e}^{\mathrm{i}\gamma}K \,|\, -I] \in \mathcal{B}^{\mathbb{C}}$, *where* $K \in \mathrm{SL}(2,\mathbb{R})$ *and* $\gamma \in \mathbb{R}$. *Denote by* $\{\sigma_n(K);\, n \in \mathbb{N}\}$ *the eigenvalues for* $\boldsymbol{U}_K$, *and by* $\{\tau_n(K);\, n \in \mathbb{N}\}$ *the eigenvalues for* $\boldsymbol{V}_K$.

i) *We always have that*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \sigma_1(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \sigma_2(K) \le \cdots . \tag{4.50}$$

ii) *If* $k_{12}k_{22} \le 0$ *and* $k_{22} \ne 0$, *then*

$$\tau_1(K) \le \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \tau_2(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \cdots ; \tag{4.51}$$

and if $k_{12}k_{22} \ge 0$ *and* $k_{12} \ne 0$, *then*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \tau_1(K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \tau_2(K) \le \cdots . \tag{4.52}$$

Note that if $K \in \mathrm{SL}(2,\mathbb{R})$, then k_{12} and k_{22} can not both equal 0, and hence either $k_{12}k_{22} \le 0$ and $k_{22} \ne 0$, or $k_{12}k_{22} \ge 0$ and $k_{12} \ne 0$. So, for each coupled self-adjoint BC, one and only one of (4.51) and (4.52) holds, unless $\lambda_n(\mathrm{e}^{\mathrm{i}\gamma}K) = \tau_n(K)$ for all $n \in \mathbb{N}$.

Theorem 4.53. *Let* $K \in \mathrm{SL}(2,\mathbb{R})$, *and introduce* $\{\sigma_n(K);\, n \in \mathbb{N}\}$ *and* $\{\tau_n(K);\, n \in \mathbb{N}\}$ *as in Theorem 4.49.*

i) *If* $k_{12} \le 0$ *and* $k_{22} > 0$, *then* $\lambda_1(K)$ *is simple, and for any* $\gamma \in (-\pi, 0) \cup (0, \pi)$, *we have that*

$$\begin{aligned} \tau_1(K) \le \lambda_1(K) < \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_1(-K) &\le \{\sigma_1(K), \tau_2(K)\} \\ \le \lambda_2(-K) < \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_2(K) &\le \{\sigma_2(K), \tau_3(K)\} \\ \le \lambda_3(K) < \lambda_3(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_3(-K) &\le \{\sigma_3(K), \tau_4(K)\} \\ \le \lambda_4(-K) < \lambda_4(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_4(K) &\le \{\sigma_4(K), \tau_5(K)\} \le \cdots . \end{aligned} \tag{4.54}$$

ii) *If* $k_{12} < 0$ *and* $k_{22} \leq 0$, *then* $\lambda_1(K)$ *is simple, and for any* $\gamma \in (-\pi, 0) \cup (0, \pi)$, *we have that*

$$\begin{aligned}
(4.55) \qquad & \lambda_1(K) < \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_1(-K) \leq \{\sigma_1(K), \tau_1(K)\} \leq \\
& \lambda_2(-K) < \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_2(K) \leq \{\sigma_2(K), \tau_2(K)\} \leq \\
& \lambda_3(K) < \lambda_3(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_3(-K) \leq \{\sigma_3(K), \tau_3(K)\} \leq \\
& \lambda_4(-K) < \lambda_4(\mathrm{e}^{\mathrm{i}\gamma}K) < \lambda_4(K) \leq \{\sigma_4(K), \tau_4(K)\} \leq \cdots .
\end{aligned}$$

iii) *If neither i) nor ii) applies to* K, *then either i) or ii) applies to* $-K$.

The inequalities of Theorem 4.53 are also generalizations of (0.3), just as those in Theorem 4.34 are. Since k_{11} and k_{12} form the first row of K and the second row of K consists of k_{21} and k_{22}, the two separated BC's $\boldsymbol{U}_K$ and $\boldsymbol{V}_K$ are easier to write down than $\boldsymbol{S}_K$ and $\boldsymbol{T}_K$.

In the following, we frequently use the fact that if $k_{12}k_{21} \geq 0$, then $k_{11}k_{22} > 0$, since $\det K = 1$.

Remark 4.56. We now compare Theorems 4.49 and 4.53 with Theorems 4.8 and 4.34. First of all,

$$\begin{aligned}
(4.57) \qquad & \boldsymbol{S}_K = \boldsymbol{U}_K \text{ if and only if } k_{12} = 0, \\
(4.58) \qquad & \boldsymbol{T}_K = \boldsymbol{U}_K \text{ if and only if } k_{11} = 0, \\
(4.59) \qquad & \boldsymbol{S}_K = \boldsymbol{V}_K \text{ if and only if } k_{22} = 0, \\
(4.60) \qquad & \boldsymbol{T}_K = \boldsymbol{V}_K \text{ if and only if } k_{21} = 0.
\end{aligned}$$

In the case of (4.57), one obtains (4.50) from (4.9); in the case of (4.58), we deduce (4.50) from (4.11); in the case of (4.59), one derives (4.52) from (4.9); while in the case of (4.60), we obtain (4.51) from (4.10) if $k_{11}k_{12} \leq 0$ (i.e., $k_{12}k_{22} \leq 0$), and (4.52) from (4.11) if $k_{11}k_{12} > 0$ (i.e., $k_{12}k_{22} > 0$).

The statements about the simplicity of $\lambda_1(K)$ in Theorem 4.53 can be deduced from those in Theorem 4.34, since $k_{11}k_{22} > 0$ when $k_{12} = 0$.

The two separated BC's $\boldsymbol{U}_K$ and $\boldsymbol{V}_K$ are equal to $\boldsymbol{S}_K$ and $\boldsymbol{T}_K$, respectively, at the same time if and only if K is diagonal. In this case, only (4.35) and (4.54) hold (replacing K by $-K$ and γ by $\gamma - \pi$ or $\gamma + \pi$ if necessary), and they are the same.

The two separated BC's $\boldsymbol{U}_K$ and $\boldsymbol{V}_K$ are equal to $\boldsymbol{T}_K$ and $\boldsymbol{S}_K$, respectively, at the same time if and only if K is off-diagonal. In this case, only (4.36) and (4.55) are true (replacing K by $-K$ and γ by $\gamma - \pi$ or $\gamma + \pi$ if necessary), and they are identical.

The inequalities in Theorems 4.8, 4.34, 4.49 and 4.53 are obtained by comparing the eigenvalues for coupled self-adjoint BC's with the eigenvalues for separated self-adjoint BC's on the natural loops. One deduces more new inequalities using comparison with the eigenvalues for self-adjoint BC's in the jump set $\mathscr{J}^{\mathbb{C}}$ on the loops. We present these inequalities next.

For each $K \in \mathrm{SL}(2,\mathbb{R})$ with $k_{11} \neq 0$, we set

$$\widehat{K} = \begin{pmatrix} k_{11} & 0 \\ k_{21} & 1/k_{11} \end{pmatrix}; \tag{4.61}$$

and for every $K \in \mathrm{SL}(2,\mathbb{R})$ with $k_{22} \neq 0$, we set

$$\widetilde{K} = \begin{pmatrix} 1/k_{22} & 0 \\ k_{21} & k_{22} \end{pmatrix}. \tag{4.62}$$

Note that $\widehat{K}, \widetilde{K} \in \mathrm{SL}(2,\mathbb{R})$. The proof of the following new inequalities is basically the same as parts of the proof of Theorem 4.8.

Theorem 4.63. *Let $[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] \in \mathcal{B}^{\mathbb{C}}$, where $K \in \mathrm{SL}(2,\mathbb{R})$ and $\gamma \in \mathbb{R}$.*
i) *If $k_{11} \neq 0$, then we have that*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}\widehat{K}) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}\widehat{K}) \le \cdots. \tag{4.64}$$

ii) *If $k_{22} \neq 0$, then we have that*

$$\lambda_1(\mathrm{e}^{\mathrm{i}\gamma}K) \le \lambda_1(\mathrm{e}^{\mathrm{i}\gamma}\widetilde{K}) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}K) \le \lambda_2(\mathrm{e}^{\mathrm{i}\gamma}\widetilde{K}) \le \cdots. \tag{4.65}$$

We remark that $[\mathrm{e}^{\mathrm{i}\gamma}\widehat{K} \mid -I]$ and $[\mathrm{e}^{\mathrm{i}\gamma}\widetilde{K} \mid -I]$ are in the jump set $\mathcal{J}^{\mathbb{C}}$.

Proof. i) Let $a_{12} = k_{12}/k_{11}$, $z = \mathrm{e}^{\mathrm{i}\gamma}/k_{11}$ and $b_{21} = k_{21}/k_{11}$; and define $\boldsymbol{A}(s)$ by (4.23). Then, $[\mathrm{e}^{\mathrm{i}\gamma}K \mid -I] = \boldsymbol{A}(a_{12})$ and $[\mathrm{e}^{\mathrm{i}\gamma}\widehat{K} \mid -I] = \boldsymbol{A}(0)$ are on the loop $\mathcal{C}_{3,z,b_{21}}$, the only separated BC on this loop is $\boldsymbol{T}_K = \lim_{s\to\pm\infty} \boldsymbol{A}(s)$, and $\mathcal{J}^{\mathbb{C}} \cap \mathcal{C}_{3,z,b_{21}} = \boldsymbol{A}(0)$. Hence, for each $n \in \mathbb{N}$, $\lambda_n(\boldsymbol{A}(s))$ is continuous in s on $(-\infty, 0)$ and $(0, +\infty)$, and (4.24) holds. Moreover, $\lambda_n(\boldsymbol{A}(s))$ is increasing in s on $(-\infty, 0)$ and $(0, +\infty)$, and we have (4.26) and (4.27). (So, for each $n \in \mathbb{N}$, the maximum value of λ_n on $\mathcal{C}_{3,z,b_{21}}$ is achieved at $\boldsymbol{A}(0)$.) Therefore, for any $s_+ \in (0, +\infty)$ and $s_- \in (-\infty, 0)$,

$$\begin{aligned} &\lambda_1(\boldsymbol{A}(s_+)) \le \nu_1(K) \le \lambda_1(\boldsymbol{A}(s_-)) \le \lambda_1(\boldsymbol{A}(0)) \le \\ &\lambda_2(\boldsymbol{A}(s_+)) \le \nu_2(K) \le \lambda_2(\boldsymbol{A}(s_-)) \le \lambda_2(\boldsymbol{A}(0)) \le \cdots, \end{aligned} \tag{4.66}$$

which implies (4.64).

ii) By similar arguments using the loop $\mathcal{C}_{4,z,b_{22}}$ with appropriate $z \in \mathbb{C}$ and $b_{22} \in \mathbb{R}$, we deduce (4.65). ∎

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Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA

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