Complex Symplecto-Geometric Characterization of Self-Adjoint Domains of Singular Differential Operators

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Abstract. The self-adjoint domains associated with a scalar differential expression of an arbitrary even order are investigated in this paper. There is a known one-to-one correspondence between these domains and the complete Lagrangian subspaces of a naturally defined symplectic space. Using this correspondence, a characterization of the self-adjoint domains is obtained for the situations where the differential expression is regular at one of the end points of its interval and in either the limit-point category or the limit-circle category at the other end point.

Key words: symplectic geometry, self-adjoint domains, differential operator, grade.

Introduction

One of the major successes of operator theory is that it offers a framework for discussing and analyzing operators in quantum mechanics. For example, the Schrödinger theory is based on properties of the differential operators in the wave equation. Most operators appearing in quantum mechanics are self-adjoint operators. Characterizing self-adjoint differential operators is a fundamental problem in operator theory, and a huge amount of effort has been devoted to it. See, for example, [3, 13, 18].

Let

$$l\left(y\right)=\sum_{k=0}^{n}\left(p_{n-k}y^{(k)}\right)^{(k)},\quad t\in[0,\infty) \tag{0.1}$$

be a $2n$-th order symmetric differential expression with real coefficients defined on $I=[0,\infty)$, where $p_{n-k}\in C^{k}\left(I\right)$, $p_{0}(t)>0$ for $t\in I$. How can one generate self-adjoint operators in L^2 associated with the symmetric differential expression? In 1976, Everitt generalized Titchmarsh-Weyl's theory to the high-order cases, and gave some self-adjoint domains with the maximum and the minimum deficiency indices [5, 6, 7], called Everitt's domains. We know that Everitt's domains are a complete solution to the problem of describing all the self-adjoint extensions of any given symmetric differential expression with the minimum deficiency index (i.e., the limit-point case); but in the limit-circle case (i.e., the maximum deficiency index case), the Everitt's domains are only special self-adjoint domains. Using the general theory of linear operators, Cao obtained a complete and direct characterization of all the self-adjoint extensions of symmetric differential expressions with the maximum deficiency index in 1985 [1], and proved that Everitt's domains are only special cases of his results. In 1986, Sun [17] extended Everitt and Cao's results to the cases where the symmetric differential expressions have middle deficiency indices. The key point of [17] is to prove a decomposition of D_m, the domain of the maximal operator L_m, such that the conditions that elements of D_m satisfy at $t = 0$ and at infinity are separated. Using this decomposition he obtained a complete characterization of all self-adjoint extensions of high-order symmetric differential operators with any deficiency indices. In subsequent years, a series of papers were accomplished by the group of Inner Mongolian University, directed by Cao, Liu and Sun, extending the results of Cao and Sun to various cases: two singular endpoints case by Shang and Zhu [16] and Li [11], self-adjoint extensions in a direct sum space by Li [12] and Fu [10], the J-self-adjoint case by Shang [14], and more. In 1990, Evans and Sobhy extended the results of [17] to more general cases [4]. All the above results come from the method of differential operators theory and functional analysis, i.e., from analysis methods.

In 1999, Everitt and Markus studied the complete description of all self-adjoint extensions of symmetric differential operators using symplectic geometry firstly [9] and [8]. They discovered a natural one-to-one correspondence between the set of all self-adjoint extensions of the minimal operator and the set of all complete Lagrangian subspaces of an associated complex symplectic space. The characterization of the self-adjoint domains amounts to the characterization of the complete Lagrangian subspaces of the complex symplectic space. Even though interesting information about self-adjoint domains is obtained from this correspondence, [9] and [8] do not have any example of a singular differential expression for which the self-adjoint domains can be explicitly characterized using this correspondence.

In this paper, using the above correspondence, we explicitly work out a characterization of the self-adjoint domains for the situations where $l(y)$ is regular at 0 and in either the limit-point category or the limit-circle category at $+\infty$. These two cases will be called the regular-limit-point case and the regular-limit-circle case, respectively. The situations where $l(y)$ is regular at $+\infty$ and in either the limit-point category or the limit-circle category at 0 can be handled similarly and hence are omitted. The characterization is parallel to the one given by Cao in [1] when $l(y)$ is in the regular-limit-circle case, and to the one obtained by Sun in [17] when $l(y)$ is in the regular-limit-point case.

The organization of this paper is as follows. In Section 1, we introduce our notation and recall some basic results. The regular-limit-point case is treated in Section 2, while the regular-limit-circle case in Section 3.

1 Preliminaries

Definition 1.1 [9] *A complex symplectic space S is a complex linear space equipped with a prescribed symplectic form $[\cdot:\cdot]$. Here, a form $[\cdot:\cdot]$: $S\times S\to\mathbb{C}$ is symplectic if*

(1) $[\cdot:\cdot]$ *is sesquilinear, i.e.,*

$$[c_1u + c_2v : w] = c_1\,[u : w] + c_2\,[v : w]\,,$$

(2) $[\cdot:\cdot]$ *is skew-Hermitian, i.e.,*

$$[u : v] = -\overline{[v : u]},\ \ so\ \ [u : c_1v + c_2w] = \bar{c}_1\,[u : v] + \bar{c}_2\,[u : w]\,,$$

(3) $[\cdot:\cdot]$ *is non-degenerate, i.e.,*

$[u:S]=0$ *implies* $u=0$,

where $u,v,w\in S$, $c_1,c_2\in\mathbb{C}$ *and* $[u:S]=\{[u:v]\,|v\in S\}$.

Definition 1.2 [9] *A linear subspace* L *in the complex symplectic space* S *is called Lagrangian in case* $[L:L]=0$, *that is,* $[u:v]=0$, *for any* $u,v\in L$. *A Lagrangian subspace* $L\subset S$ *is complete if* $u\in S$ *and* $[u:L]=0$ *imply* $u\in L$.

Definition 1.3 [9] *Let* S *be a complex symplectic space. Then, linear subspaces* S_- *and* S_+ *are symplectic ortho-complements in* S *if*

(1) $S=\operatorname{span}\{S_-,S_+\}$,

(2) $[S_-:S_+]=0$.

In this case, we write $S=S_-\oplus S_+$.

We define the maximal operator and minimal operator associated with $l(y)$ as follows:

$$T_{max}(y)=l(y),$$
$$D(T_{max})=\{y\in L^2(I)|\,y^{[k]}\in AC(I)\text{ for }k=0,1,2,\cdots,2n-1,y^{[2n]}\in L^2(I)\};$$
$$T_{min}(y)=l(y),$$
$$D(T_{min})=\{y\in D(T_{max})|y^{[k]}(0)=0\ \text{ for }\ k=0,\cdots,2n-1,\ [y,z](\infty)=0$$
$$\text{for any }z\in D(T_{max})\},$$

where $[y,z]$ is the Langrange bilinear form associated with $l(y)$, and $y^{[k]}$ is the kth quasi-derivative of y defined by

$$y^{[k]}=\frac{\mathrm{d}^k y}{\mathrm{d}x^k},\ \ 0\leq k\leq n-1,$$
$$y^{[n]}=p_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n},$$
$$y^{[n+k]}=p_k\frac{\mathrm{d}^{n-k}y}{\mathrm{d}x^{n-k}}-\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{[n+k-1]}\right),\ \ 1\leq k\leq n.$$

By the theory of ordinary differential operators, T_{min} and T_{max} are closed operators, $T^*_{max}=T_{min}$, $T^*_{min}=T_{max}$ and T_{min} is a symmetric operator. Set

$$S=D\left(T_{max}\right)/D\left(T_{min}\right),$$

and define a symplectic form on S by

$$[\mathbf{f}:\mathbf{g}]=\left[f+D\left(T_{min}\right):g+D\left(T_{min}\right)\right]=[f,g]_0^\infty,$$

where $\mathbf{f}=f+D\left(T_{min}\right),\ \mathbf{g}=g+D\left(T_{min}\right)\in S$.

We have

$$D(T_{min})=\{f\in D(T_{max})|[f:D(T_{max})]=0\}.$$

Lemma 1.1 [9] *$S = D(T_{max})/D(T_{min})$ equipped with $[\cdot : \cdot]$ is a complex symplectic space.*

Lemma 1.2 [9] *$S = S_- \oplus S_+$, where $S_+ = \{\mathbf{f} \in S | f^{[k]}(0) = 0,\ k = 0, 1, \cdots, 2n-1\}$, $S_- = \{\mathbf{f} \in S |$ for any $g \in D(T_{max})$, $[f, g](\infty) = 0\}$.*

Lemma 1.3 [9]*(Balanced intersection principle) Let L be a complete Lagrangian subspace, then*

$$0 \leq \frac{1}{2}\dim S_- - \dim L \cap S_- = \frac{1}{2}\dim S_+ - \dim L \cap S_+ \leq \frac{1}{2}\min\{\dim S_-, \dim S_+\}.$$

Definition 1.4 [9] *Let L be a complete Lagrangian subspace, if*

$$k = \frac{1}{2}\dim S_- - \dim L \cap S_- = \frac{1}{2}\dim S_+ - \dim L \cap S_+,$$

then L is called of the k-th grade, $D(T_L)$ is also called of the k-th grade.

Remark 1.1 *A similar number of "grade" was introduced in 1996 by Shang [15] who tried to classify the J-self-adjoint domains of J-symmetric differential expressions in the limit-circle case.*

Lemma 1.4 [9]*(GKN Theorem)*

(1) *There exists a self-adjoint extension of T_{min} if and only if there exists a complete Lagrangian subspace $L \subset S$.*

(2) *The Lagrangian subspace $L \subset S$ is complete if and only if $\dim L = \frac{1}{2}\dim S$.*

(3) *there exists a natural one-to-one correspondence between the set $\{T\}$ of all self-adjoint extensions T of T_0 and the set $\{L_T\}$ of all complete Lagrangian subspaces $L_T \subset S$ such that*

$$L_T = D(T)/D(T_{min}).$$

(4) *Let $L \subset S$ be a complete Lagrangian subspace. Then, the unique self-adjoint extension T_L corresponding to L is given by*

$$D(T_L) = c_1 f_1 + \ldots + c_m f_m + D(T_{min}),$$

where $\{\mathbf{f_1}, \mathbf{f_2}, \cdots, \mathbf{f_m}\}$ is a basis for L with $f_1, \cdots, f_m \in D(T_{max})$, and $c_1, c_2, \cdots, c_m$ are complex numbers.

Remark 1.2 *From Lemma 1.4, we know that characterizing the self-adjoint extensions of T_{min} is equivalent to the problem of discussing the complete Lagrangian subspaces of $S = D_{(max)}/D_{(min)}$, i.e., the description and classification of complete Lagrangian subspaces of S is equivalent to the description and classification of the self-adjoint domains of $l(y)$.*

2 Regular-Limit-Point Case

Throughout this section, let $l(y)$ be in the regular-limit-point case. Recall that in this case, $l(y)$ is regular at 0 and belongs to the limit-point category at $+\infty$.

Lemma 2.1 *Let $l(y)$ be in the regular-limit-point case, then*

$$D(T_{min}) = \{f \in D(T_{max}) | f^{[k]}(0) = 0, \;\; k = 0, 1, \cdots, 2n-1\}.$$

Proof Since $l\,(y)$ is in the limit-point case and the deficiency indices are (n, n), we have $[f, g]\,(\infty) = 0$ for any $\forall f, g \in D\,(T_{max})\,.$ ■

We have

Lemma 2.2 $S_- = S$, $S_+ = \{0\}$.

Lemma 2.3 $\dim S = \dim D\,(T_{max})\,/D\,(T_{min}) = 2n$.

Theorem 2.1 [8] *In the regular-limit-point case, all complete Lagrangian subspaces are of the 0-th grade.*

Because $\dim S = 2n$, S and $\mathbb{C}^{2n}$ are isomorphic as complex vectorspaces, let $e^1 = (1, 0, \cdots, 0)\,,\, e^2 = (0, 1, 0, \cdots, 0)\,, \ldots, e^{2n} = (0, \cdots, 0, 1)$ be the usual basis for $\mathbb{C}^{2n}$, then $S = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$. For $\mathbf{f} \in S$, we choose the coordinates of $\mathbf{f}$ as:

$$\mathbf{f} = (f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0)) = f(0)e^1 + f^{[1]}(0)e^2 + \cdots + f^{[2n-1]}(0)e^{2n}. \tag{2.1}$$

Theorem 2.2 *For any $\mathbf{f}, \mathbf{g} \in S$, $[\mathbf{f} : \mathbf{g}] = \mathbf{f}H\mathbf{g}^*$ where*

$$H = \begin{pmatrix} 0 & -H_1 \\ H_1 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}.$$

Proof Since $l\,(y)$ is in the regular-limit-point case, by the definition of the symplectic form of S, we have

$$[\mathbf{f} : \mathbf{g}] = [f, g]_0^\infty = -\,[f, g]\,(0) = \mathbf{f}H\mathbf{g}^*.$$ ■

Theorem 2.3 *L is a 0-th grade complete Lagrangian subspace of S if and only if there exist $a_{ij} \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 2n$, such that*

$$L = \text{span}\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_n E\} \tag{2.2}$$

and

(1) $\text{rank}\, A = n$,

(2) $\alpha_i H \alpha_j^* = 0, 1 \le i, j \le n$,

where $A = (a_{ij})_{n \times 2n}$, $\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{i,2n})$, $i = 1, 2, \cdots, n$, and $E = (e^1, e^2, \cdots, e^{2n})^T$.

Proof ($\Leftarrow$) For any $\mathbf{f}, \mathbf{g} \in L$, there exist $s_i, t_i \in \mathbb{C}$, $i = 1, 2, \cdots, n$, such that

$$\begin{aligned}\mathbf{f} &= s_1\alpha_1 E + s_2\alpha_2 E + \cdots + s_n\alpha_n E \\ &= (\sum_{i=1}^{n} s_i a_{i1})e^1 + (\sum_{i=1}^{n} s_i a_{i2})e^2 + \cdots + (\sum_{i=1}^{n} s_i a_{i,2n})e^{2n},\end{aligned}$$

$$\begin{aligned}\mathbf{g} &= t_1\alpha_1 E + t_2\alpha_2 E + \cdots + t_n\alpha_n E \\ &= (\sum_{i=1}^{n} t_i a_{i1})e^1 + (\sum_{i=1}^{n} t_i a_{i2})e^2 + \cdots + (\sum_{i=1}^{n} t_i a_{i,2n})e^{2n}.\end{aligned}$$

From Theorem 2.2 and (2), we get

$$[\mathbf{f} : \mathbf{g}] = (\sum_{i=1}^{n} s_i a_{i1}, \sum_{i=1}^{n} s_i a_{i2}, \cdots, \sum_{i=1}^{n} s_i a_{i,2n})H(\sum_{i=1}^{n} t_i a_{i1}, \sum_{i=1}^{n} t_i a_{i2}, \cdots, \sum_{i=1}^{n} t_i a_{i,2n})^* = 0.$$

So, $[L : L] = 0$, that is, L is a Lagrangian subspace of S. From (1), $\dim L = n$, so L is a 0-th grade complete Lagrangian subspace of S.

($\Rightarrow$) If L is a 0-th grade complete Lagrangian subspace of S, then $\dim L = n$, $\dim L \cap S_+ = 0$, $\dim L \cap S_- = n$ and $[L : L] = 0$. So, there exist $a_{ij} \in \mathbb{C}$, $i = 1, \cdots, n$, $j = 1, 2, \cdots, 2n$, such that (2.2) holds and (1) and (2) are satisfied. ■

Corollary 2.1 *L is a 0-th grade complete Lagrangian subspace of S if and only if there exist $a_{ij} \in \mathbb{C}$, $i = 1, 2, \cdots, n, j = 1, 2, \ldots, 2n$, such that*

$$L = \{\mathbf{f} \in S \mid \exists s_i \in \mathbb{C}, i = 1, \cdots, n, (f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0))^T = A^T(s_1, s_2, \cdots, s_n)^T\}$$

and

(1) $\operatorname{rank} A = n$,

(2) $\alpha_i H \alpha_j^* = 0$, $1 \le i, j \le n$,

where $A = (a_{ij})_{n\times 2n}, \alpha_i = (a_{i1}, a_{i2}, \cdots, a_{i,2n}), i = 1, 2 \cdots, n.$

3 Regular-Limit-Circle Case

Throughout this section, let $l(y)$ be in the regular-limit-circle case. Recall that in this case, $l(y)$ is regular at 0 and belongs to the limit-circle category at $+\infty$. Then, the deficiency indices of $l(y)$ are $(2n, 2n)$, and there exists a linear independent system of solutions φ_1, ..., φ_{2n} of $l(y) = 0$ satisfying $([\varphi_i, \varphi_j](0)) = J$, where

$$J = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix}.$$

Lemma 3.1 *[2] For any $f, g \in D(T_{max})$, we have*

$$[f, g](x) = i\sum_{r=1}^{n} [f, \varphi_r](x)[g, \overline{\varphi}_r](x) - i\sum_{r=n+1}^{2n} [f, \varphi_r](x)[g, \overline{\varphi}_r](x).$$

By the definition of the maximal and minimal operator domains, Lemma 2.1 and Lemma 3.1, we have

Lemma 3.2 *If all solutions of* $l(y) = 0$ *belong to* $L^2(I)$, *then*

$$D(T_{min}) = \{f \in D(T_{max}) \big| f^{[k]}(0) = 0,\ [f, \varphi_{k+1}](\infty) = 0,\ k = 0, 1, \cdots, 2n-1\}.$$

Lemma 3.3 $\dim S = 4n$, $\dim S_- = \dim S_+ = 2n$.

Theorem 3.1 *In the regular-limit-circle case, there exists a k-th grade* $(k = 0, 1, \cdots, n)$ *complete Lagrangian subspace of* S, *and the grade* k *of any complete Lagrangian subspace is among* $0, 1, \cdots, n$.

Since $\dim S = 4n$, S and $\mathbb{C}^{4n}$ are linear isomorphic. Let

$$e^1 = (1, 0, 0, \cdots, 0),\ e^2 = (0, 1, 0, \cdots, 0),\ \cdots,\ e^{2n} = (\overbrace{0, \cdots, 0, 1}^{2n}, 0, \cdots, 0),$$

$$f^1 = (\overbrace{0, \cdots, 0, 1}^{2n+1}, 0, \cdots, 0), f^2 = (\overbrace{0, \cdots, 0, 1}^{2n+2}, 0, \cdots, 0), \cdots,\ f^{2n} = (0, \cdots, 0, 1).$$

Then $S = \text{span}\{e^1, e^2, \cdots, e^{2n}, f^1, f^2, \cdots, f^{2n}\}$.

For $\mathbf{f} \in S$, we choose the coordinates of $\mathbf{f}$ as:

$$\begin{aligned} \mathbf{f} &= (f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0), [f, \varphi_1](\infty), [f, \varphi_2](\infty), \cdots, [f, \varphi_{2n}](\infty)) \\ &= f(0)e^1 + f^{[1]}(0)e^2 + \cdots + f^{[2n-1]}(0)e^{2n} \\ &\quad + [f, \varphi_1](\infty)f^1 + [f, \varphi_2](\infty)f^2 + \cdots + [f, \varphi_{2n}](\infty)f^{2n}. \end{aligned} \tag{3.1}$$

Theorem 3.2 *For any* $\mathbf{f}, \mathbf{g} \in S$, $[\mathbf{f} : \mathbf{g}] = \mathbf{f}H\mathbf{g}^*$, *where*

$$H = \begin{pmatrix} 0 & -H_1 & 0 & 0 \\ H_1 & 0 & 0 & 0 \\ 0 & 0 & iI_n & 0 \\ 0 & 0 & 0 & -iI_n \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}_{n\times n},$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}.$$

Proof By the definition of the symplectic form and Lemma 3.1, we have

$$\begin{aligned} [\mathbf{f} : \mathbf{g}] &= [f, g]_0^\infty = [f, g](\infty) - [f, g](0) \\ &= \{i \sum_{r=1}^{n} [f, \varphi_r](\infty)[g, \overline{\varphi}_r](\infty) - i \sum_{r=n+1}^{2n} [f, \varphi_r](\infty)[g, \overline{\varphi}_r](\infty)\} \\ &\quad - \sum_{k=1}^{n} \{f^{[k-1]}(0)\overline{g}^{[2n-k]}(0) - f^{[2n-k]}(0)\overline{g}^{[k-1]}(0)\} \\ &= \mathbf{f}H\mathbf{g}^*. \end{aligned}$$

■

Theorem 3.3 $S_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$, $S_+ = \text{span}\{f^1, f^2, \cdots, f^{2n}\}$.

Proof First we prove that $S_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$. For any $\mathbf{f} \in S_-$, we have $f \in D(T_{max})$ and $[f, \varphi_k](\infty) = 0$, $k = 1, 2, \cdots, 2n$. We get

$$\mathbf{f} = f(0)e^1 + f^{[1]}(0)e^2 + \cdots + f^{[2n-1]}(0)e^{2n} \in \text{span}\{e^1, e^2, \cdots, e^{2n}\}.$$

Thus, $S_- \subseteq \text{span}\{e^1, e^2, \cdots, e^{2n}\}$. If $\mathbf{f} \in \text{span}\{e^1, e^2, \cdots, e^{2n}\}$, then

$$\mathbf{f} = f(0)e^1 + f^{[1]}(0)e^2 + \cdots + f^{[2n-1]}(0)e^{2n},$$

i.e., $[f, \varphi_k](\infty) = 0$, $k = 1, 2, \cdots, 2n$. So, $\text{span}\{e^1, e^2, \cdots, e^{2n}\} \subseteq S_-$. Therefore, $S_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$.

Similarly, we get $S_+ = \text{span}\{f^1, f^2, \cdots, f^{2n}\}$. ■

Theorem 3.4 *L is a 0-th grade complete Lagrangian subspace of S if and only if there exist a_{ij}, $b_{ij} \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 2n$, such that*

$$L = \text{span}\left\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_{2n} E\right\} \tag{3.2}$$

and

(1) $\text{rank}\, A_n = \text{rank}\, B_n = n$,

(2) $\alpha_i H \alpha_j^* = 0$, $1 \leq i, j \leq 2n$,

where $A_n = (a_{ij})_{n\times 2n}$, $B_n = (b_{ij})_{n\times 2n}$, $E = (e^1, e^2, \cdots, e^{2n}, f^1, f^2, \cdots, f^{2n})^T$ *and*

$$\alpha_i = \begin{cases} (a_{i1}, \cdots, a_{i,2n}, 0, \cdots, 0), & 1 \leq i \leq n, \\ (0, \cdots, 0, b_{i-n,1}, \cdots, b_{i-n,2n}), & n+1 \leq i \leq 2n. \end{cases}$$

Proof ($\Leftarrow$) For any $\mathbf{f}, \mathbf{g} \in L$, there exist $s_{1i}, s_{2i}, t_{1i}, t_{2i} \in \mathbb{C}$, $i = 1, 2, \cdots, n$, such that

$$\mathbf{f} = \sum_{i=1}^{n} s_{1i}\alpha_i E + \sum_{i=1}^{n} t_{1i}\alpha_{n+i} E = (\sum_{i=1}^{n} s_{1i}a_{i1}, \cdots, \sum_{i=1}^{n} s_{1i}a_{i,2n}, \sum_{i=1}^{n} t_{1i}b_{i1}, \cdots, \sum_{i=1}^{n} t_{1i}b_{i,2n})E,$$

$$\mathbf{g} = \sum_{i=1}^{n} s_{2i}\alpha_i E + \sum_{i=1}^{n} t_{2i}\alpha_{n+i} E = (\sum_{i=1}^{n} s_{2i}a_{i1}, \cdots, \sum_{i=1}^{n} s_{2i}a_{i,2n}, \sum_{i=1}^{n} t_{2i}b_{i1}, \cdots, \sum_{i=1}^{n} t_{2i}b_{i,2n})E.$$

By Theorem 3.2 and (2), we have

$$\begin{aligned} [\mathbf{f} : \mathbf{g}] &= (\sum_{i=1}^{n} s_{1i}a_{i1}, \cdots, \sum_{i=1}^{n} s_{1i}a_{i,2n}, \sum_{i=1}^{n} t_{1i}b_{i1}, \cdots, \sum_{i=1}^{n} t_{1i}b_{i,2n})H \\ &\quad \cdot(\sum_{i=1}^{n} s_{2i}a_{i1}, \cdots, \sum_{i=1}^{n} s_{2i}a_{i,2n}, \sum_{i=1}^{n} t_{2i}b_{i1}, \cdots, \sum_{i=1}^{n} t_{2i}b_{i,2n})^* = 0. \end{aligned}$$

So, $[L : L] = 0$, i.e., L is a Lagrangian subspace. From (1), $\dim L = 2n$. By Lemma 1.3, we have L is a complete Lagrangian subspace of S, and

$$\dim L \cap S_- = \dim L \cap S_+ = n.$$

Therefore, L is of the 0-th grade.

($\Rightarrow$) Since L is a 0-th grade complete Lagrangian subspace, we have

$$\dim L = 2n, \quad \dim L \cap S_- = \dim L \cap S_+ = n, \quad [L : L] = 0,$$

and there exist $a_{ij}, b_{ij} \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots 2n$, such that

$$L = \text{span}\left\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_{2n} E\right\}.$$

It is easy to prove that (1) and (2) are satisfied. ■

Corollary 3.1 *L is a 0-th grade complete Lagrangian subspace of S if and only if there exist $a_{ij}, b_{ij} \in \mathbb{C}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 2n$, such that*

$$\begin{aligned} L &= \{\mathbf{f} \in S | \, \exists s_i, t_i \in \mathbb{C}, \; i = 1, 2, \cdots, n, \\ &\quad (f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0))^T = A_n^T (s_1, s_2, \cdots, s_n)^T, \\ &\quad ([f, \varphi_1](\infty), [f, \varphi_2](\infty), \cdots, [f, \varphi_{2n}](\infty))^T = B_n^T (t_1, t_2, \cdots, t_n)^T\}, \end{aligned}$$

and

(1) $\text{rank}\, A_n = \text{rank}\, B_n = n$;

(2) $\alpha_i H \alpha_j^*, \;\; 1 \le i, j \le 2n$,

where $A_n = (a_{ij})_{n\times 2n}, B_n = (b_{ij})_{n \times 2n}$, and

$$\alpha_i = \begin{cases} (a_{ij}, \cdots, a_{i,2n}, 0, \cdots, 0), & 1 \le i \le n \\ (0, \cdots, 0, b_{i-n,1}, \cdots, b_{i-n,2n}), & n+1 \le i \le 2n. \end{cases}$$

Example 3.1 *$L = \text{span}\,\{e^1 + e^2, e^3 + e^4, \cdots, e^{2n-1} + e^{2n}, f^1 + f^2, f^3 + f^4, \cdots, f^{2n-1} + f^{2n}\} = \left\{\mathbf{f} \in S | f^{[k]}(0) = f^{[k+1]}(0), [f, \varphi_{k+1}](\infty) = [f, \varphi_{k+2}](\infty), \;\; k = 0, 1, \cdots, 2n-2\right\}$ is a 0-th grade complete Lagrangian subspace.*

Theorem 3.5 *L is a first grade complete Lagrangian subspace of S if and only if there exist $a_{ij}, b_{ij}, c_{kj}, d_{kj} \in \mathbb{C}$, $i = 1, 2, \cdots, n-1$, $j = 1, 2, \cdots, 2n$, $k = 1, 2$, such that*

$$L = \text{span}\left\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_{2n} E\right\}$$

and

(1) $\text{rank}\, A_{n+1} = \text{rank}\, B_{n+1} = n + 1$,

(2) $\alpha_i H \alpha_j^* = 0, \;\; 1 \le i, j \le 2n$,

where E is defined in Theorem 3.4, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,2n} \\ c_{11} & c_{12} & \cdots & c_{1,2n} \\ c_{21} & c_{22} & \cdots & c_{2,2n} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,2n} \\ d_{11} & d_{12} & \cdots & d_{1,2n} \\ d_{21} & d_{22} & \cdots & d_{2,2n} \end{pmatrix},$$

$$\alpha_i = \begin{cases} (a_{i1}, \cdots, a_{i,2n}, 0, \cdots, 0), & 1 \le i \le n-1, \\ (0, \cdots, 0, b_{i-n+1,1}, \cdots, b_{i-n+1,2n}), & n \le i \le 2n-2, \\ (c_{i-2n+2,1}, \cdots, c_{i-2n+2,2n}, d_{i-2n+2,1}, \cdots, d_{i-2n+2,2n}), & 2n-1 \le i \le 2n. \end{cases}$$

Corollary 3.2 *L is a k-th grade complete Lagrangian subspace of S if and only if there exist $a_{ij}, b_{ij}, c_{lj}, d_{lj} \in \mathbb{C}$, $i = 1, 2, \cdots, n-k$, $j = 1, 2, \cdots, 2n$, $l = 1, 2, \cdots, 2k$, such that*

$$L = \operatorname{span}\left\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_{2n} E\right\},$$

and

(1) $\operatorname{rank} A_{n+k} = \operatorname{rank} B_{n+k} = n + k,$

(2) $\alpha_i H \alpha_j^* = 0, \quad 1 \le i, j \le 2n,$

where E is defined in Theorem 3.4, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,2n} \\ c_{11} & c_{12} & \cdots & c_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{2k,1} & c_{2k,2} & \cdots & c_{2k,2n} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n-k,1} & b_{n-k,2} & \cdots & b_{n-k,2n} \\ d_{11} & d_{12} & \cdots & d_{1,2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{2k,1} & d_{2k,2} & \cdots & d_{2k,2n} \end{pmatrix},$$

$$\alpha_i = \begin{cases} (a_{i1}, \cdots, a_{i,2n}, 0, \cdots, 0), \quad 1 \le i \le n-k, \\ (0, \cdots, 0, b_{i-n+k,1}, \cdots, b_{i-n+k,2n}), \quad n-k+1 \le i \le 2n-2k, \\ (c_{i-2n+2k,1}, \cdots, c_{i-2n+2k,2n}, d_{i-2n+2k,1}, \cdots, d_{i-2n+2k,2n}), \quad 2n-2k+1 \le i \le 2n. \end{cases}$$

Corollary 3.3 *L is a k-th grade complete Lagrangian subspace of S if and only if there exist $a_{ij}, b_{ij}, c_{lj}, d_{lj} \in \mathbb{C}$, $i = 1, 2, \cdots, n-k$, $j = 1, 2, \cdots, 2n$, $l = 1, 2, \cdots, 2k$, such that*

$$\begin{aligned} L \;=\; & \{\mathbf{f} \in S | \, \exists s_i, t_i, u_j \in \mathbb{C}, \ i = 1, 2, \cdots, n-k, \ j = 1, 2, \cdots, 2k, \\ & (f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0))^T = A_{n+k}^T (s_1, s_2, \cdots, s_{n-k}, u_1, u_2, \cdots, u_{2k})^T, \\ & ([f, \varphi_1](\infty), [f, \varphi_2](\infty), \cdots, [f, \varphi_{2n}](\infty))^T = B_{n+k}^T (t_1, t_2, \cdots, t_{n-k}, u_1, u_2, \cdots, u_{2k})^T\}. \end{aligned}$$

and

(1) $\operatorname{rank} A_{n+k} = \operatorname{rank} B_{n+k} = n + k,$

(2) $\alpha_i H \alpha_j^* = 0, \quad 1 \le i, j \le 2n,$

where A, B and $\alpha_1, \cdots, \alpha_{2n}$ are defined in Corollary 3.2.

Theorem 3.6 *L is an n-th grade complete Lagrangian subspace of S if and only if there exist $c_{kj}, d_{kj} \in \mathbb{C}$, $k = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, 2n$, such that*

$$L = \operatorname{span}\left\{\alpha_1 E, \alpha_2 E, \cdots, \alpha_{2n} E\right\},$$

and

(1) $\operatorname{rank} A_{2n} = \operatorname{rank} B_{2n} = 2n,$

(2) $\alpha_i H \alpha_j^* = 0, \quad 1 \le i, j \le 2n,$

where $A_{2n} = (c_{sj})_{2n\times 2n}\,, B_{2n} = (d_{sj})_{2n\times 2n}\,, \alpha_i = (c_{i1}, \cdots, c_{i,2n}, d_{i1}, \cdots, d_{i,2n})\,,\ i = 1, 2, \cdots,$ $2n$, *and* E *is defined in Theorem 3.4.*

Corollary 3.4 *L is an n-th grade complete Lagrangian subspace of S if and only if there exist $c_{kj}, d_{kj} \in \mathbb{C}$, $j = 1, 2, \cdots, 2n$, $k = 1, 2, \cdots, 2n$, such that*

$$\begin{aligned} L &= \{\mathbf{f} \in S | (A_{2n}^T)^{-1}(f(0), f^{[1]}(0), \cdots, f^{[2n-1]}(0))^T = \\ &\qquad (B_{2n}^T)^{-1}([f, \varphi_1](\infty), [f, \varphi_2](\infty), \cdots, [f, \varphi_{2n}](\infty))^T\}, \end{aligned}$$

and

(1) $\operatorname{rank} A_{2n} = \operatorname{rank} B_{2n} = 2n$,

(2) $\alpha_i H \alpha_j^* = 0, \quad 1 \le i, j \le 2n$,

where $A_{2n} = (c_{sj})_{2n\times 2n}$, $B_{2n} = (d_{sj})_{2n\times 2n}$, *and* $\alpha_i = (c_{i1}, \cdots, c_{i,2n}, d_{i1}, \cdots, d_{i,2n})$ *for* $i = 1, 2, \cdots, 2n$.

Example 3.2 $L = \operatorname{span}\left\{e^1 + f^1, e^2 + f^2, \cdots, e^{2n} + f^{2n}\right\} = \{\mathbf{f} \in S \big| f^{[k]}(0) = [f, \varphi_{k+1}](\infty),$ $k = 0, 1, \cdots, 2n-1\}$ *is an n-th grade complete Lagrangian subspace of S.*

Remark 3.1 *From the above analysis, there are $n+1$ classes of complete Lagrangian subspaces of S, they are all described by boundary conditions, and only in the case of 0-th grade complete Lagrangian subspaces, the boundary conditions are separated. Other boundary conditions are coupled.*

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