Completeness of Eigenfunctions of Sturm-Liouville Problems with Transmission Conditions

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Abstract. In this paper, we investigate a class of Sturm-Liouville problems with eigenparameter-dependent boundary conditions and transmission conditions at an interior point. A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of such a problem coincide with those of A. We show that the eigenvalues of the problem are analytically simple, and the eigenfunctions of A are complete in H.

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Introduction

The Sturm-Liouville theory plays an important role in solving many problems in mathematical physics. It is an active area of research in pure and applied mathematics. In recent years, there has been a growing interest in Sturm-Liouville problems (SLP's) with eigenparameter-dependent boundary conditions, i.e., the eigenparameter appears not only in the differential equations of the SLP's but also in the boundary conditions of the problems [6, 9]. There is a vast amount of literature on this subject, see [2, 3, 4, 5, 8, 12], etc. Moreover, some boundary value problems which may have discontinuities in the solution or its derivative at an interior point c are also studied. Conditions are imposed on the left and right limits of solutions and their derivatives at an interior point c and are often called "transmission conditions" or "interface conditions". These problems often arise in varied assortment of physical transfer problems, see [9, 11, 12]. Also, some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness). In this class of problems, transmission conditions across the interfaces should be added since the plate is laminated. The study of the structure of the solution in the matching region of the layer with the basis solution in the plate leads to consideration of an eigenvalue problem for a second-order differential operator with piecewise continuous coefficients and transmission conditions [14].

In this paper, we consider a class of SLP's with eigenparameter-dependent boundary conditions and transmission conditions, i.e., study regular Sturm-Liouville equation

$$lu := -(a(x)u'(x))' + q(x)u(x) = \lambda u(x) \quad \text{on } I, \tag{0.1}$$

where $I = [-1, 0) \cup (0, 1]$, $a(x) = {a_1}^2$ for $x \in [-1, 0)$ and $a(x) = {a_2}^2$ for $x \in (0, 1]$, a_1, a_2 are nonzero real constants, $q(x) \in L^1(I, \mathbb{R})$, and $\lambda \in \mathbb{C}$ is the so-called eigenparameter;

with the boundary condition

$$l_1 u := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \tag{0.2}$$

the eigenparameter-dependent boundary condition

$$l_2 u := \lambda(\beta_1' u(1) - \beta_2' u'(1)) + \beta_1 u(1) - \beta_2 u'(1) = 0, \tag{0.3}$$

and the transmission conditions

$$l_3 u := u(0+) - \alpha_3 u(0-) - \beta_3 u'(0-) = 0, \tag{0.4}$$

$$l_4 u := u'(0+) - \alpha_4 u(0-) - \beta_4 u'(0-) = 0, \tag{0.5}$$

where the coefficients α_i, β_i and β_j' ($i = 1, ..., 4$ and $j = 1, 2$) are real numbers. Throughout this paper, we assume that

$$\theta = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} > 0, \qquad \rho = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0,$$

and $\alpha_1^2 + \alpha_2^2 \neq 0$.

SLP's with transmission conditions have been studied by many authors. In [1, 5], Mukhtarov, Demirci, etc gave asymptotic formulas for eigenvalues and the corresponding eigenfunctions for these problems. In [10], the complete descriptions of self-adjoint boundary conditions of the Schrödinger operator with $\delta(x)$ or $\delta'(x)$ interaction are given. Adjoint and self-adjoint boundary value problems with interface conditions have been studied by Zettl in [16]. Such problems with point interactions are also studied in [7], etc.

In this paper, we also deal with the class of problems (0.1)–(0.5), by means of a combination of the methods of [1] and [2]. In Section 1, a self-adjoint linear operator A (not only symmetric [1]) is defined in a suitable Hilbert space H such that the eigenvalues of the problem (0.1)–(0.5) coincide with those of A. In Section 2, we prove that the eigenvalues of the problem are analytically simple. Finally, in Section 3, we prove that the eigenfunctions of A are complete in H. Note that each eigenfunction of A consists of an eigenfunction of the original problem and a real number.

1 Operator Formulation

The relation between a symmetric linear operator A defined in a suitable Hilbert space H and the problem (0.1)–(0.5) has been introduced in [1]. Here, we repeat the definition and prove that the operator A is self-adjoint, not only symmetric.

We define the inner product in $L^2(I)$ as

$$\langle f, g\rangle_1 = \frac{1}{a_1^2}\int_{-1}^{0} f_1\overline{g}_1 + \frac{1}{a_2^2\theta}\int_{0}^{1} f_2\overline{g}_2, \quad \forall f, g \in L^2(I),$$

where $f_1(x) = f(x)|_{(-1,0)}$ and $f_2(x) = f(x)|_{(0,1)}$. It is easy to verify that $(L^2(I), \langle\cdot,\cdot\rangle_1)$ is a Hilbert space. For simplicity, it is denoted by H_1.

The inner product in $H := H_1 \oplus \mathbb{C}$ is defined by

$$\langle F, G\rangle = \langle f, g\rangle_1 + \frac{1}{\rho\theta} h\overline{k}$$

for

$$F=(f(x),h),\ \ G=(g(x),k)\in H,$$

where $f,g\in H_1,\ \ h,k\in\mathbb{C}$.

We define the operator A in H as follows:

$$\begin{aligned}\mathcal{D}(A)=&\{(f(x),h)\in H\big|f_1,f_1'\in AC_{loc}((-1,0)),f_2,f_2'\in AC_{loc}((0,1)),\\ &lf\in H_1,\ l_1f=l_3f=l_4f=0,h=\beta_1'f(1)-\beta_2'f'(1)\},\end{aligned}$$

$$AF=(lf,-(\beta_1f(1)-\beta_2f'(1)))\quad \text{for } F=(f,\beta_1'f(1)-\beta_2'f'(1))\in\mathcal{D}(A).$$

Note that by our assumption on $q(x)$ and Theorem 3.2 in [15], for each $(f,h)\in\mathcal{D}(A)$, $f_1,\ f_1'$ are continuous on $[-1,0]$, and $f_2,\ f_2'$ are continuous on $[0,1]$. For simplicity, for $(f,h)\in\mathcal{D}(A)$, set

$$N(f)=\beta_1f(1)-\beta_2f'(1),\ \ N'(f)=\beta_1'f(1)-\beta_2'f'(1).$$

So, we can study the problem (0.1)–(0.5) in H by considering the operator equation $AF=\lambda F$. Obviously, we have

Lemma 1.1 *The eigenvalues of the boundary value problem* (0.1)–(0.5) *coincide with those of A, and its eigenfunctions are the first components of the corresponding eigenfunctions of A.*

Lemma 1.2 *The domain $\mathcal{D}(A)$ is dense in H.*

Proof Suppose that $F\in H$ is orthogonal to all $G\in\mathcal{D}(A)$ with respect to the inner product $\langle\cdot,\cdot\rangle$, where $F=(f(x),h),\ G=(g(x),k)$. Let $\widetilde{C}_0^\infty$ denote the set of functions

$$\phi(x)=\begin{cases}\varphi_1(x), & x\in[-1,0);\\ \varphi_2(x), & x\in(0,1],\end{cases}$$

where $\varphi_1(x)\in C_0^\infty[-1,0)$ and $\varphi_2(x)\in C_0^\infty(0,1]$. Since $\widetilde{C}_0^\infty\oplus 0\subset\mathcal{D}(A)$ $(0\in\mathbb{C})$, any $U=(u(x),0)\in\widetilde{C}_0^\infty\oplus 0$ is orthogonal to F, namely,

$$\langle F,U\rangle=\frac{1}{a_1^2}\int_{-1}^0 f(x)\overline{u(x)}dx+\frac{1}{a_2^2\theta}\int_0^1 f(x)\overline{u(x)}dx=\langle f,u\rangle_1.$$

This implies that $f(x)$ is orthogonal to $\widetilde{C}_0^\infty$ in H_1 and hence vanishes. So, $\langle F,G\rangle=\frac{1}{\rho\theta}h\bar{k}=0$. Thus, $h=0$ since $k=N'(g)$ can be chosen arbitrarily. So, $F=(0,0)$. Hence, $\mathcal{D}(A)$ is dense in H. ■

Theorem 1.1 *The linear operator A is self-adjoint in H.*

Proof For all $F,G\in\mathcal{D}(A)$, (0.2) implies that $f(-1)\bar{g}'(-1)-f'(-1)\bar{g}(-1)=0$, and direct calculations using (0.4) and (0.5) then yield that

$$\begin{aligned}\langle AF,G\rangle=&\langle F,AG\rangle+W(f,\bar{g};0-)-W(f,\bar{g};-1)+\frac{1}{\theta}W(f,\bar{g};1)-\frac{1}{\theta}W(f,\bar{g};0+)\\ &-\frac{1}{\rho\theta}(N(f)\overline{N'(g)}-N'(f)\overline{N(g)})=\langle F,AG\rangle,\end{aligned}$$

where $W(f,g;x)$ denotes the Wronskians $f(x)g'(x)-f'(x)g(x)$. So, A is symmetric.

It remains to show that if $\langle AF,W\rangle=\langle F,U\rangle$ for all $F=(f,N'(f))\in\mathcal{D}(A)$, then $W\in\mathcal{D}(A)$ and $AW=U$, where $W=(w(x),h)$ and $U=(u(x),k)$, i.e., (i) $w_1,w_1'\in AC_{loc}((-1,0))$, $w_2,w_2'\in AC_{loc}((0,1))$ and $lw\in H_1$; (ii) $h=N'(w)=\beta_1'w(1)-\beta_2'w'(1)$; (iii) $l_1w=l_3w=l_4w=0$; (iv) $u(x)=lw$; (v) $k=-N(w)=-\beta_1w(1)+\beta_2w'(1)$.

For all $F\in\widetilde{C}_0^\infty\oplus 0\subset\mathcal{D}(A)$, we obtain

$$\frac{1}{a_1^2}\int_{-1}^{0}(lf)\bar{w}\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1(lf)\bar{w}\,\mathrm{d}x=\frac{1}{a_1^2}\int_{-1}^{0}f\bar{u}\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1 f\bar{u}\,\mathrm{d}x,$$

namely, $\langle lf,w\rangle_1=\langle f,u\rangle_1$. Hence, by standard Sturm–Liouville theory, (i) and (iv) hold. By (iv), the equation $\langle AF,W\rangle=\langle F,U\rangle$, $\forall F\in\mathcal{D}(A)$, becomes

$$\begin{aligned}&\frac{1}{a_1^2}\int_{-1}^{0}(lf)\bar{w}\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1(lf)\bar{w}\,\mathrm{d}x+\frac{-N(f)\bar{h}}{\rho\,\theta}\\&=\frac{1}{a_1^2}\int_{-1}^{0}f(l\bar{w})\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1 f(l\bar{w})\,\mathrm{d}x+\frac{N'(f)\bar{k}}{\rho\,\theta}.\end{aligned}$$

So,

$$\langle lf,w\rangle_1=\langle f,lw\rangle_1+\frac{N'(f)\bar{k}}{\rho\,\theta}+\frac{N(f)\bar{h}}{\rho\,\theta}.$$

However,

$$\begin{aligned}\langle lf,w\rangle_1=&\frac{1}{a_1^2}\int_{-1}^{0}(-a_1^2f''+q(x)f)\bar{w}\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1(-a_2^2f''+q(x)f)\bar{w}\,\mathrm{d}x\\=&\frac{1}{a_1^2}\int_{-1}^{0}f(l\bar{w})\,\mathrm{d}x+\frac{1}{a_2^2\theta}\int_0^1 f(l\bar{w})\,\mathrm{d}x+W(f,\bar{w};0-)-W(f,\bar{w};-1)+\\&\frac{1}{\theta}W(f,\bar{w};1)-\frac{1}{\theta}W(f,\bar{w};0+)\\=&\langle f,lw\rangle_1+W(f,\bar{w};0-)-W(f,\bar{w};-1)+\frac{1}{\theta}W(f,\bar{w};1)-\frac{1}{\theta}W(f,\bar{w};0+).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{N'(f)\bar{k}}{\rho\,\theta}+\frac{N(f)\bar{h}}{\rho\,\theta}=&W(f,\bar{w};0-)-W(f,\bar{w};-1)+\frac{1}{\theta}W(f,\bar{w};1)-\frac{1}{\theta}W(f,\bar{w};0+)\\=&(f(0-)\bar{w}'(0-)-f'(0-)\bar{w}(0-))-(f(-1)\bar{w}'(-1)-f'(-1)\bar{w}(-1))+\\&\frac{1}{\theta}(f(1)\bar{w}'(1)-f'(1)\bar{w}(1))-\frac{1}{\theta}(f(0+)\bar{w}'(0+)-f'(0+)\bar{w}(0+)).\end{aligned}\tag{1.6}$$

By Naimark's Patching Lemma [13], there is an $F\in\mathcal{D}(A)$ such that $f(-1)=f'(-1)=f(0-)=f'(0-)=f(0+)=f'(0+)=0$, $f(1)=\beta_2'$ and $f'(1)=\beta_1'$. For such an F, $N'(f)=0$. Thus, from (1.6) we obtain $h=\beta_1'w(1)-\beta_2'w'(1)$. Namely, (ii) holds. Similarly, one proves (v).

It remains to show that (iii) holds. Choose $F\in\mathcal{D}(A)$ so that $f(1)=f'(1)=f(0-)=f'(0-)=0$, $f(-1)=\alpha_2$ and $f'(-1)=-\alpha_1$. Then $N'(f)=N(f)=0$. From (1.6), we get $\alpha_1w(-1)+\alpha_2w'(-1)=0$. Let $F\in\mathcal{D}(A)$ satisfy $f(1)=f'(1)=f(-1)=f'(-1)=$

$f(0+)=0$, $f(0-)=-\beta_3$, $f'(0-)=\alpha_3$ and $f'(0+)=\theta$. Then $N(f)=N'(f)=0$. By (1.6), we have $w(0+)=\alpha_3 w(0-)+\beta_3 w'(0-)$. Finally, choose $F\in\mathcal{D}(A)$ so that $f(1)=f'(1)=f(-1)=f'(-1)=f'(0+)=0$, $f(0-)=\beta_4$, $f'(0-)=-\alpha_4$ and $f(0+)=\theta$. Then $N(f)=N'(f)=0$. From (1.6), we obtain $w'(0+)=\alpha_4 w(0-)+\beta_4 w'(0-)$. ■

Corollary 1.1 *The eigenvalues of* (0.1)–(0.5) *are real, and if* λ_1 *and* λ_2 *are two different eigenvalues of* (0.1)–(0.5)*, then the corresponding eigenfunctions* $f(x)$ *and* $g(x)$ *are orthogonal in the sense of*

$$\frac{1}{a_1^2}\int_{-1}^{0} f\overline{g}+\frac{1}{a_2^2\theta}\int_0^1 f\overline{g}+\frac{1}{\rho\,\theta}(\beta_1' f(1)-\beta_2' f'(1))(\beta_1'\overline{g}(1)-\beta_2'\overline{g}'(1))=0.$$

2 Simplicity of Eigenvalues

We consider the initial-value problem

$$\begin{cases} -a_1^2 u''(x)+q(x)u(x)=\lambda u(x), & x\in[-1,0),\\ u(-1)=\alpha_2, \quad u'(-1)=-\alpha_1. \end{cases}$$

In terms of existence and uniqueness in ordinary differential equation theory, the initial-value problem has a unique solution $\varphi_1(x,\lambda)$ for every $\lambda\in\mathbb{C}$. Similarly, the initial-value problem

$$\begin{cases} -a_2^2 u''(x)+q(x)u(x)=\lambda u(x), & x\in(0,1],\\ u(0)=\alpha_3\varphi_1(0,\lambda)+\beta_3\varphi_1'(0,\lambda),\\ u'(0)=\alpha_4\varphi_1(0,\lambda)+\beta_4\varphi_1'(0,\lambda) \end{cases}$$

has a unique solution $\varphi_2(x,\lambda)$. For each given $x\in[-1,0)$, $\varphi_1(x,\lambda)$ is an entire function of λ; for every $x\in(0,1]$, $\varphi_2(x,\lambda)$ is an entire function of λ .

Now define a function $\phi(x,\lambda)$ on $x\in[-1,0)\cup(0,1]$ by

$$\phi(x,\lambda)=\begin{cases} \varphi_1(x,\lambda), & x\in[-1,0);\\ \varphi_2(x,\lambda), & x\in(0,1]. \end{cases}$$

Obviously, $\phi(x,\lambda)$ satisfies (0.1), (0.2), (0.4) and (0.5). Similarly, we define the function

$$\chi(x,\lambda)=\begin{cases} \chi_1(x,\lambda), & x\in[-1,0);\\ \chi_2(x,\lambda), & x\in(0,1], \end{cases}$$

which satisfies (0.1), (0.3), (0.4) and (0.5).

The Wronskian $W(\varphi_i(x,\lambda),\chi_i(x,\lambda))$ $(i=1,2)$ are independent of the variable x. Let $w_i(\lambda)=W(\varphi_i(x,\lambda),\chi_i(x,\lambda))$ and $w(\lambda)=w_1(\lambda)$, and then we obtain $w_2(\lambda)=\theta w(\lambda)$.

Lemma 2.1 *[1] The eigenvalues of* (0.1)–(0.5) *coincide with the zeros of the entire function* $w(\lambda)$.

Definition 2.1 *The analytic multiplicity of an eigenvalue* λ *of* (0.1)–(0.5) *is its order as a root of the characteristic equation* $w(\lambda)=0$. *The geometric multiplicity of an eigenvalue is the dimension of its eigenspace, i.e., the number of its linearly independent eigenfunctions.*

For convenience, set $\phi=\phi(x,\lambda)$, $\chi_{1\lambda}=\frac{\partial\chi_1}{\partial\lambda}$, $\chi'_{1\lambda}=\frac{\partial\chi'_1}{\partial\lambda}$, etc.

Theorem 2.1 *The eigenvalues of* (0.1)–(0.5) *are analytically simple.*

Proof Let $\lambda = u + iv$. We differentiate the equation $l\chi = \lambda\chi$ with respect to λ and have

$$l\chi_{\lambda} = \lambda\chi_{\lambda} + \chi.$$

By integration by parts, we get

$$\langle l\chi_{\lambda}, \phi\rangle_1 - \langle \chi_{\lambda}, l\phi\rangle_1 = (\chi_{1\lambda}\bar{\varphi}_1' - \chi_{1\lambda}'\bar{\varphi}_1)\big|_{-1}^{0} + \frac{1}{\theta}(\chi_{2\lambda}\bar{\varphi}_2' - \chi_{2\lambda}'\bar{\varphi}_2)\big|_{0}^{1}. \tag{2.7}$$

Substituting $l\chi_{\lambda} = \lambda\chi_{\lambda} + \chi$ and $l\phi = \lambda\phi$ into the left side of (2.7), we have

$$\lambda\langle \chi_{\lambda}, \phi\rangle_1 + \langle \chi, \phi\rangle_1 - \langle \chi_{\lambda}, \lambda\phi\rangle_1 = \langle \chi, \phi\rangle_1 + 2iv\langle \chi_{\lambda}, \phi\rangle_1.$$

Moreover,

$$\begin{aligned}
&(\chi_{1\lambda}\bar{\varphi}_1' - \chi_{1\lambda}'\bar{\varphi}_1)\big|_{-1}^{0} + \frac{1}{\theta}(\chi_{2\lambda}\bar{\varphi}_2' - \chi_{2\lambda}'\bar{\varphi}_2)\big|_{0}^{1}\\
=&\chi_{1\lambda}(0,\lambda)\bar{\varphi}_1'(0,\lambda) - \chi_{1\lambda}'(0,\lambda)\bar{\varphi}_1(0,\lambda) - \chi_{1\lambda}(-1,\lambda)\bar{\varphi}_1'(-1,\lambda) + \chi_{1\lambda}'(-1,\lambda)\bar{\varphi}_1(-1,\lambda)\\
+&\frac{1}{\theta}(\chi_{2\lambda}(1,\lambda)\bar{\varphi}_2'(1,\lambda) - \chi_{2\lambda}'(1,\lambda)\bar{\varphi}_2(1,\lambda)) - \frac{1}{\theta}(\chi_{2\lambda}(0,\lambda)\bar{\varphi}_2'(0,\lambda) - \chi_{2\lambda}'(0,\lambda)\bar{\varphi}_2(0,\lambda))\\
=&\alpha_1\chi_{1\lambda}(-1,\lambda) + \alpha_2\chi_{1\lambda}'(-1,\lambda) + \chi_{1\lambda}(0,\lambda)\bar{\varphi}_1'(0,\lambda) - \chi_{1\lambda}'(0,\lambda)\bar{\varphi}_1(0,\lambda) +\\
&\frac{1}{\theta}(\beta_2'\bar{\varphi}_2'(1,\lambda) - \beta_1'\bar{\varphi}_2(1,\lambda)) - \frac{1}{\theta}(\chi_{2\lambda}(0,\lambda)\bar{\varphi}_2'(0,\lambda) - \chi_{2\lambda}'(0,\lambda)\bar{\varphi}_2(0,\lambda))\\
=&\alpha_1\chi_{1\lambda}(-1,\lambda) + \alpha_2\chi_{1\lambda}'(-1,\lambda) + \frac{1}{\theta}(\beta_2'\bar{\varphi}_2'(1,\lambda) - \beta_1'\bar{\varphi}_2(1,\lambda)).
\end{aligned}$$

Note that

$$w'(\lambda) = \alpha_2\chi_{1\lambda}'(-1,\lambda) + \alpha_1\chi_{1\lambda}(-1,\lambda).$$

So, (2.7) becomes

$$w'(\lambda) = \langle \chi, \phi\rangle_1 + 2iv\langle \chi_{\lambda}, \phi\rangle_1 - \frac{1}{\theta}(\beta_2'\bar{\varphi}_2'(1,\lambda) - \beta_1'\bar{\varphi}_2(1,\lambda)). \tag{2.8}$$

Now we consider the simplicity of the eigenvalues of (0.1)–(0.5). Let μ be arbitrary zero of $w(\lambda)$. By Corollary 1.1, μ is real. Since

$$w(\mu) = \begin{vmatrix} \varphi_1(x,\mu) & \chi_1(x,\mu) \\ \varphi_1'(x,\mu) & \chi_1'(x,\mu) \end{vmatrix} = 0,$$

we have $\varphi_1(x,\mu) = c_1\chi_1(x,\mu)$ $(c_1 \neq 0)$ and $\varphi_2(x,\mu) = c_2\chi_2(x,\mu)$ $(c_2 \neq 0)$, where $c_1, c_2 \in \mathbb{C}$. From

$$\begin{aligned}
\varphi_2(0,\mu) &= c_1(\alpha_3\chi_1(0,\mu) + \beta_3\chi_1'(0,\mu)) = c_1\chi_2(0,\mu),\\
\varphi_2'(0,\mu) &= c_1(\alpha_4\chi_1(0,\mu) + \beta_4\chi_1'(0,\mu)) = c_1\chi_2'(0,\mu),
\end{aligned}$$

we get $c_1 = c_2 \neq 0$. Thus, simple calculations using (2.8) and the initial values of χ_2 at $x = 1$ give

$$w'(\mu) = \bar{c}_1\Big(\frac{1}{a_1^2}\int_{-1}^{0}\big|\chi_1(x,\mu)\big|^2\,\mathrm{d}x + \frac{1}{a_2^2\theta}\int_{0}^{1}\big|\chi_2(x,\mu)\big|^2\,\mathrm{d}x + \frac{\rho}{\theta}\Big).$$

Note that $\rho > 0\,, \theta > 0$ and $\bar{c}_1 \neq 0$, so $w'(\mu) \neq 0$. Hence, the analytic multiplicity of μ is one. By Lemma 2.1, the proof is completed. ■

Theorem 2.2 *All eigenvalues of* (0.1)–(0.5) *are also geometrically simple.*

Proof If f and g are two eigenfunctions for an eigenvalue λ_* of (0.1)–(0.5), then (0.2) implies that $f(-1) = Cg(-1)$ and $f'(-1) = Cg'(-1)$ for some constant $C \in \mathbb{R}$. By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (0.4) and (0.5), we have that $f = Cg$ on $[-1, 0]$ and on $[0, 1]$. Thus, the geometric multiplicity of λ_* is one. ∎

3 Completeness of Eigenfunctions

Theorem 3.1 *The operator* A *has only point spectrum, i.e.,* $\sigma(A) = \sigma_p(A)$.

Proof It suffices to prove that if γ is not an eigenvalue of A, then $\gamma \in \rho(A)$. Since A is self-adjoint, we only consider a real γ. We investigate the equation $(A - \gamma)Y = F \in H$, where $F = (f, h)$.

Consider the initial-value problem

$$\begin{cases} ly - \gamma y = f, \quad x \in I, \\ \alpha_1 y(-1) + \alpha_2 y'(-1) = 0; \\ y(0+) = \alpha_3 y(0-) + \beta_3 y'(0-); \\ y'(0+) = \alpha_4 y(0-) + \beta_4 y'(0-). \end{cases} \tag{3.9}$$

Let $u(x)$ be the solution of the equation $lu - \gamma u = 0$ satisfying

$$u(-1) = \alpha_2, \quad u'(-1) = -\alpha_1;$$

$$u(0+) = \alpha_3 u(0-) + \beta_3 u'(0-);$$

$$u'(0+) = \alpha_4 u(0-) + \beta_4 u'(0-).$$

In fact

$$u(x) = \begin{cases} u_1(x), & x \in [-1, 0); \\ u_2(x), & x \in (0, 1], \end{cases}$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\begin{cases} -a_1^2 u'' + q(x)u = \gamma u, & x \in [-1, 0); \\ u(-1) = \alpha_2, \quad u'(-1) = -\alpha_1, \end{cases} \tag{3.10}$$

and $u_2(x)$ is the unique solution of the problem

$$\begin{cases} -a_2^2 u'' + q(x)u = \gamma u, & x \in (0, 1]; \\ u_2(0) = \alpha_3 u_1(0) + \beta_3 u_1'(0); \\ u_2'(0) = \alpha_4 u_1(0) + \beta_4 u_1'(0). \end{cases}$$

Let

$$w(x) = \begin{cases} w_1(x), & x \in [-1, 0), \\ w_2(x), & x \in (0, 1] \end{cases}$$

be a solution of $lw - \gamma w = f$ satisfying

$$\alpha_1 w(-1) + \alpha_2 w'(-1) = 0;$$

$$w(0+) = \alpha_3 w(0-) + \beta_3 w'(0-);$$
$$w'(0+) = \alpha_4 w(0-) + \beta_4 w'(0-).$$

Then, (3.9) has the general solution

$$y(x) = \begin{cases} du_1 + w_1, & x \in [-1, 0); \\ du_2 + w_2, & x \in (0, 1], \end{cases} \tag{3.11}$$

where $d \in \mathbb{C}$.

Since γ is not an eigenvalue of (0.1)–(0.5), we have

$$\gamma(\beta_1' u_2(1) - \beta_2' u_2'(1)) + (\beta_1 u_2(1) - \beta_2 u_2'(1)) \neq 0. \tag{3.12}$$

The second component of $(A - \gamma)Y = F$ involves the equation

$$-N(y) - \gamma N'(y) = h,$$

namely,

$$\beta_2 y'(1) - \beta_1 y(1) - \gamma(\beta_1' y(1) - \beta_2' y'(1)) = h. \tag{3.13}$$

Substituting (3.11) into (3.13), we get

$$(\beta_2 u_2'(1) - \beta_1 u_2(1) + \gamma\beta_2' u_2'(1) - \gamma\beta_1' u_2(1))d = h + \beta_1 w_2(1) - \beta_2 w_2'(1) + \gamma\beta_1' w_2(1) - \gamma\beta_2' w_2'(1).$$

In view of (3.12), we know that d is uniquely solvable. Hence, y is uniquely determined.

The above arguments show that $(A - \gamma I)^{-1}$ is defined on all of H. We obtain that $(A - \gamma I)^{-1}$ is bounded by Theorem 1.1 and the Closed Graph Theorem. Thus, $\gamma \in \rho(A)$. Hence, $\sigma(A) = \sigma_p(A)$. ■

Lemma 3.1 *[1] The eigenvalues of the boundary value problem* (0.1)–(0.5) *are bounded below, and they are countably infinite and can cluster only at* ∞.

For every $\delta \in \mathbb{R} \backslash \sigma_p(A)$, we have the following immediate conclusion.

Lemma 3.2 *Let* λ *be an eigenvalue of* $A - \delta I$*, and* V *a corresponding eigenfunction. Then,* $\frac{1}{\lambda}$ *is an eigenvalue of* $(A - \delta I)^{-1}$*, and* V *is a corresponding eigenfunction. The converse is also true.*

On the other hand, if μ is an eigenvalue of A and U is a corresponding eigenfunction, then $\mu - \delta$ is an eigenvalue of $A - \delta I$, and U is a corresponding eigenfunction. The converse is also true. Accordingly, the discussion about the completeness of the eigenfunctions of A is equivalent to considering the corresponding property of $(A - \delta I)^{-1}$.

By Lemma 1.1, Lemma 3.1 and Corollary 1.1, we suppose that $\{\lambda_n; n \in \mathbb{N}\}$ is the sequence (real sequence, of course) of eigenvalues of A, then $\{\lambda_n - \delta; n \in \mathbb{N}\}$ is the sequence of eigenvalues of $A - \delta I$. We may assume that

$$|\lambda_1 - \delta| \leq |\lambda_2 - \delta| \leq \cdots \leq |\lambda_n - \delta| \leq \cdots \to \infty.$$

Let $\{\mu_n; n \in \mathbb{N}\}$ be the sequence of eigenvalues of $(A - \delta I)^{-1}$. Then, $\mu_n = \frac{1}{\lambda_n - \delta}$ and

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n| \geq \cdots \to 0.$$

Note that, 0 is not an eigenvalue of $(A - \delta I)^{-1}$.

Theorem 3.2 *The operator A has compact resolvents, i.e., for each $\delta \in \mathbb{R} \setminus \sigma(A) = \mathbb{R} \setminus \sigma_p(A)$, $(A - \delta I)^{-1}$ is compact on H.*

Proof Let $\{\mu_1, \mu_2, \ldots\}$ be the eigenvalues of $(A - \delta I)^{-1}$, and $\{P_1, P_2, \ldots\}$ the orthogonal projections of finite rank onto the corresponding eigenspaces. Since $\{\mu_1, \mu_2, \ldots\}$ is a bounded sequence and all P_n's are mutually orthogonal, we have $\sum_{n=1}^{\infty} \mu_n P_n$ is strongly convergent to the bounded operator $(A - \delta I)^{-1}$, i.e., $(A - \delta I)^{-1} = \sum_{n=1}^{\infty} \mu_n P_n$. Because for every $\alpha > 0$, the number of μ_n's satisfying $|\mu_n| > \alpha$ is finite, and all P_n's are of finite rank, we obtain that $(A - \delta I)^{-1}$ is compact. ■

In terms of the above statements and the spectral theorem for compact operator, we obtain the following theorem.

Theorem 3.3 *The eigenfunctions of the problem* (0.1)–(0.5)*, augmented to become eigenfunctions of A, are complete in H, i.e., if we let $\{\Phi_n = (\phi_n(x), N'(\phi_n)); n \in \mathbb{N}\}$ be a maximum set of orthonormal eigenfunctions of A, where $\{\phi_n(x); n \in \mathbb{N}\}$ are eigenfunctions of* (0.1)–(0.5)*, then for all $F \in H$, $F = \sum_{n=1}^{\infty} \langle F, \Phi_n \rangle \Phi_n$.*

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