The Self-adjoint Extensions of Singular Differential Operators with a Real Regularity Point

Aiping Wang and Jiong Sun
Department of Mathematics, Inner Mongolia University, Hohhot 010021, China
E-mail: masunimu.edu.cn

Abstract. This paper deals with the self-adjoint domains of a singular symmetric differential expression $l(y)$ with a middle deficiency index, under the condition that $\Pi(L_0) \cap \mathbb{R} \neq \emptyset$, where $\Pi(L_0)$ is the regularity domain of the correspoinding minimal operator L_0. A complete analytic description of the self-adjoint domains of $l(y)$ is obtained by giving a new decomposition of the maximal operator domain D_M using the L^2-solutions of the equation $l(y) = \lambda_0 y$ with $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$. The description is independent of the properties of $l(y)$ at infinity, the singular point of $l(y)$.

Key words: symmetric differential operator; self-adjoint extensions; regularity domain; deficiency indices.

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Introduction

The description of self-adjoint domains of symmetric differential expression $l(y)$ is a fundamental problem in the theory of differential operators. It has a profound background in physics and applied mathematics. In 1960's, Everitt and his collaborators generalized the Titchmarsh-Weyl methods, and gave some results on self-adjoint domains in some special cases (e.g., the limit circle case and the limit point case [5]). Afterwards, using the theory of linear operators, Cao and Sun [1, 2, 3, 9] obtained a complete and direct characterization of all self-adjoint extensions of symmetric differential operators. In particular, Sun gave a new decomposition of the maximal operator domain of $l(y)$ of order n. Using the decomposition, a complete description of the self-adjoint extensions of the minimal operators L_0 generated by $l(y)$ was given; and it is the first such description in the case of middle deficiency indices. In 1990, Evans and Ibrahim [4] generalized the result to more general differential expressions M, and gave a characterization of all the regularly solvable operators and their adjoints generated by M. By Cao and Sun's methods, Shang [8] got the J-self-adjoint extensions of a J-symmetric $l(y)$, where J denotes complex conjugation. Fu [6] extended Sun's result to the characterization of self-adjoint extensions of differential operators in direct sum spaces. In this century, Wang and Sun [10] gave a complex symplecto-geometric description of J-symmetric differential operators. However, we notice that the determination of boundary conditions depends on the properties of the solutions of $l(y) = \lambda y$ at infinity, the singular point of $l(y)$. So, it is difficult to realize the self-adjoint extensions.

In this paper, under the condition that $\Pi(L_0) \cap \mathbb{R} \neq \emptyset$, we use the solutions of $l(y) = \lambda_0 y$ with $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$ and Cao's [2] and Sun's [9] methods to give a complete analytic description of self-adjoint domains of $l(y)$ with middle deficiency indices. This result improves the conclusion of [9], and the determination of boundary conditions is independent of the properties of $l(y)$ at infinity.

1 Preliminaries

Definition 1.1 *Let T be a linear operator in a Hilbert space H. The set*

$$\Pi(T) = \{z \in \mathbb{C} : \textit{there exists a constant } k(z) > 0 \textit{such that} \parallel (z - T)f \parallel \geq k(z) \parallel f \parallel \\ \textit{for all } f \in D(T)\}$$

is called the regularity domain of T.

Lemma 1.1 *[7] Let T be a linear operator in a Hilbert space H, then $\Pi(T)$ is open.*

Definition 1.2 *Let λ be a complex number. The subspace $R(\lambda - T)^\perp$ is called the defect space of T and λ. The cardinal number $n_\lambda = dim R(\lambda - T)^\perp$ is called the deficiency index of T and λ.*

Lemma 1.2 *[7] If T is a symmetric operator, then $\mathbb{C}\backslash\mathbb{R} \subseteq \Pi(T)$, and the deficiency index of T is constant on each connected subset of $\Pi(T)$.*

Let n_+ and n_- denote the deficiency indices of the symmetric operator T associated with the upper and lower half-planes, respectively. The pair (n_+, n_-) are called the deficiency indices of T. Denote $\operatorname{def} T = (n_+, n_-)$.

Lemma 1.3 *Let T be a symmetric operator in a complex Hilbert space. If $\Pi(T) \cap \mathbb{R} \neq \emptyset$, then T has self-adjoint extensions.*

Proof Since $\Pi(T) \cap \mathbb{R} \neq \emptyset$, by Lemma 1.1, we have $\Pi(T)$ is connected. By Lemma 1.2, we get $n_+ = n_-$; and hence T has equal deficiency indices. Therefore, T has self-adjoint extensions. ■

From Definition 1.2 and Lemma 1.2, we have

Lemma 1.4 *If $\lambda_0 \in \Pi(T) \cap \mathbb{R}$, and n_{λ_0} is the deficiency index of T and λ_0, then* $\operatorname{def} T = (n_{\lambda_0}, n_{\lambda_0})$.

Throughout this paper, we assume that

$$l(y) = \sum_{j=0}^{n} p_{n-j}(t) y^{(j)}, \quad t \in [0, \infty),$$

is a singular symmetric differential expression of order n with equal deficiency indices (m, m), where $p_0(t), p_1(t), \cdots, p_n(t)$ are complex functions satisfying suitable differentiable conditions on $[0, \infty)$, and $[\frac{n+1}{2}] \leq m \leq n$ with $[\frac{n+1}{2}]$ denoting the integer part of

$\frac{n+1}{2}$. Let L_M and L_0 denote the maximal operator and the minimal operator defined by the differential expression $l(y)$ restricted to the sets D_M and D_0, respectively. For any matrix A, we denote its transpose by A^T and its complex conjugate transpose by A^*.

Let $[\cdot,\cdot]$ denote the Langrange bilinear form associated with $l(y)$: for all $y, z \in D_M$,

$$[y,z](t) = \sum_{i=1}^{n} \sum_{j+k=i-1} (-1)^j y^{(k)} (p_{n-m}\bar{z})^{(j)} = \sum_{j,k=1}^{n} q_{jk}(t) y^{(k-1)} \bar{z}^{(j-1)}.$$

Set $Q(t) = (q_{jk}(t))_{n\times n}$, and call $Q(t)$ the matrix of Langrange bilinear form associated with $l(y)$. By short calculations, we obtain $\det Q(t) = (p_0(x))^n \neq 0$, i.e., $Q(t)$ is non-singular.

Suppose $\Pi(L_0) \cap \mathbb{R} \neq \emptyset$. Let $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$. Because the deficiency indices of $l(y)$ are (m,m), by Lemma 1.4, we obtain that $l(y) = \lambda_0 y$ has exactly m linearly independent square integrable solutions on $[0,\infty)$.

Let $z_1, \cdots, z_n$ be functions in D_M satisfying the following conditions:

$$\begin{aligned} z_i^{(k-1)}(0) &= \delta_{ik}, \quad z_i^{(k-1)}(1) = 0 \ \text{ for } k, i = 1, \cdots, n, \\ z_i(t) &= 0 \quad\quad \text{for } i = 1, ..., n, \ t \geq 1, \end{aligned} \tag{1.1}$$

where δ_{ik} is the Kronecker delta (for the existence of these functions, see Sect. 17 Lemma 2 in [7]).

Theorem 1.1 *Let $\lambda_0 \in \Pi(L_0) \cap \mathbb{R}$. Then, the equation $l(y) = \lambda_0 y$ has $2m-n$ linearly independent square integrable solutions $\theta_1, \cdots, \theta_{2m-n}$ on $[0,\infty)$ satisfying*

$$\begin{gathered} \operatorname{rank} E = 2m-n, \\ D_M = D_0 \dotplus \operatorname{span}\{z_1, \cdots, z_n\} \dotplus \operatorname{span}\{\theta_1, \cdots, \theta_{2m-n}\}, \end{gathered} \tag{1.2}$$

where

$$E = ([\theta_i, \theta_j](0))_{1\leq i,j \leq 2m-n}. \tag{1.3}$$

Proof From the proof of Theorem 1 of [9], we have

$$D_M = D_0 \dotplus \operatorname{span}\{z_1, \cdots z_n\} \dotplus \operatorname{span}\{\varphi_1, \cdots, \varphi_{2m-n}\}, \tag{1.4}$$

where φ_1, ..., $\varphi_{2m-n} \in L^2[0,\infty)$ are the solutions of $l(y) = \lambda y$ or $l(y) = \bar{\lambda} y$ with $Im(\lambda) \neq 0$ such that $\operatorname{rank}([\varphi_i, \varphi_j](\infty))_{1\leqslant i,j \leqslant 2m-n} = 2m-n$. Because $l(y) = \lambda_0 y$ has m linearly independent square integrable solutions $\theta_1, \cdots, \theta_m$ on $[0,\infty)$, by(1.4), we have

$$\theta_i = y_{0i} + \sum_{s=1}^{n} d_{is} z_s + \sum_{j=1}^{2m-n} c_{ij} \varphi_j, \tag{1.5}$$

where each $y_{0i} \in D_0$. Since $z_s(t) = 0$ for $t \geqslant 1$ and $[y_{0i}, \theta_i](\infty) = 0$, we obtain

$$([\theta_k, \theta_l](\infty))_{1\leqslant k,l \leqslant m} \;=\; \Big([\sum_{j=1}^{2m-n} c_{kj}\varphi_j, \sum_{j=1}^{2m-n} c_{lj}\varphi_j](\infty)\Big)_{1\leqslant k,l\leqslant m} = C([\varphi_i, \varphi_j](\infty))_{1\leqslant i,j\leqslant 2m-n} C^*,$$

where $C = (c_{ij})_{m\times(2m-n)}$. Hence,

$$\operatorname{rank}([\theta_k, \theta_l](\infty))_{1\leqslant k,l\leqslant m} \leqslant \operatorname{rank}([\varphi_i, \varphi_j](\infty))_{1\leqslant i,j\leqslant 2m-n} = 2m-n.$$

Since $l(\theta_i) = \lambda_0\theta_i$ and $\lambda_0 \in \mathbb{R}$, from Green's formula, we have

$$[\theta_k, \theta_l](\infty) = [\theta_k, \theta_l](0), \quad k, l = 1, \cdots, m. \tag{1.6}$$

So,

$$([\theta_k, \theta_l](\infty))^T_{m\times m} = ([\theta_k, \theta_l](0))^T_{m\times m} = (R(\bar{\theta}_l)(0)Q(0)(R(\theta_k)(0))^T)^T_{m\times m} = \Theta^*(0)Q(0)\Theta(0)\ ,$$

where $R(\theta_k)(0) = (\theta_k(0), \theta_k'(0), \cdots, \theta_k^{(n-1)}(0))$, and $\Theta(t)_{n\times m}$ denotes the Wronski matrix of $\{\theta_i(t);\ i = 1, \cdots, m\}$. Since $\operatorname{rank} Q(0) = n$ and $\operatorname{rank}\Theta(0) = \operatorname{rank}\Theta^*(0) = m$, it follows that $\operatorname{rank}([\theta_k, \theta_l](\infty))_{m\times m} \geqslant 2m - n$. Here we have used the fact that $\operatorname{rank}(MN) \geqslant \operatorname{rank} M + \operatorname{rank} N - n$ for any matrices $M = M_{u\times n}$ and $N = N_{n\times v}$. Hence,

$$\operatorname{rank}([\theta_k, \theta_l](\infty))_{m\times m} = \operatorname{rank}([\theta_k, \theta_l](0))_{m\times m} = 2m - n. \tag{1.7}$$

Then, by $\overline{C}([\varphi_i, \varphi_j](\infty))^T_{1\leqslant i,j\leqslant 2m-n}C^T = ([\theta_k, \theta_l](\infty))^T_{m\times m} = \Theta^*(0)Q(0)\Theta(0)$, we know $\operatorname{rank} C \geqslant 2m - n$. Since $\operatorname{rank} C_{m\times(2m-n)} \leqslant 2m - n$, we have $\operatorname{rank} C = 2m - n$. Without loss of generality, we can assume that the first $2m-n$ rows of C are linearly independent. Denote $C_1 = (c_{ij})_{1\leqslant i,j\leqslant 2m-n}$, then $\operatorname{rank} C_1 = 2m - n$. Using (1.5), we have

$$([\theta_i, \theta_j](\infty))_{1\leqslant i,j\leqslant 2m-n} = C_1([\varphi_i, \varphi_j](\infty))_{1\leqslant i,j\leqslant 2m-n}C_1^*.$$

So, $\operatorname{rank}([\theta_i, \theta_j](\infty))_{1\leqslant i,j\leqslant 2m-n} = 2m - n$. Therefore, there exist $2m - n$ linearly independent solutions, say $\theta_1, \cdots, \theta_{2m-n}$, of $l(y) = \lambda_0 y$ such that $\operatorname{rank} E = 2m - n$. It remains to show that each $y \in D_M$ can be uniquely written as the following form:

$$y = y_0 + \sum_{s=1}^{n} d_s z_s + \sum_{j=1}^{2m-n} \tau_j\theta_j\ ,$$

where $y_0 \in D_0$. From the equation

$$\theta_i = y_{0i} + \sum_{s=1}^{n} d_{is}z_s + \sum_{j=1}^{2m-n} c_{ij}\varphi_j, \quad i = 1, \cdots, 2m - n, \tag{1.8}$$

and $\operatorname{rank} C_1 = 2m - n$, we can solve for each φ_j and obtain the unique representation

$$\varphi_j = \widetilde{y}_{0j} + \sum_{i=1}^{n} \widetilde{c}_{ji}z_i + \sum_{s=1}^{2m-n} \widetilde{b}_{js}\theta_s\ ,$$

where $\widetilde{y}_{0j} \in D_0$. Using the method similar to Theorem 1 of [9], we can prove (1.2). The proof is completed. ■

Lemma 1.5 *[7] Let the deficiency indices of L_0 be (m, m). Then, a linear subspace D in D_M is the domain of a self-adjoint extension of L_0 if and only if there exist functions $v_1, \cdots, v_m$ in D_M which satisfy*

(i) $v_1, \cdots, v_m$ are linearly independent modulo D_0;

(ii) $[v_i, v_j]_0^\infty = 0$ $(i, j = 1, \cdots, m)$,

and

(iii) $D = \{y \in D_M | [y, v_j]_0^\infty = 0,\ j = 1, \cdots, m\}$.

2 The Main Result

Let

$$B_1 = ([\varphi_i, \varphi_j](\infty))^T_{1\leqslant i,j\leqslant 2m-n}, \tag{2.1}$$

where $\varphi_1, \cdots, \varphi_{2m-n}$ are defined in the proof of Theorem 1.1.

Lemma 2.1 *[9] Let $l(y)$ be a singular symmetric ordinary differential expression of order n with equal deficiency indices (m, m), where $[\frac{n+1}{2}] \leqslant m \leqslant n$. Then, a linear subspace D in D_M is the domain of a self-adjoint extension of L_0 if and only if there exist an $m \times n$ matrix M_1 and an $m \times (2m-n)$ matrix N_1 satisfying*

$(1')$ $\operatorname{rank}(M_1 \oplus N_1) = m$,

$(2')M_1Q^{-1}(0)M_1^* + N_1B_1N_1^* = 0$,

and such that

$$(3')\ D = \{y \in D_M : M_1 \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N_1 \begin{pmatrix} [y, \varphi_1](\infty) \\ \vdots \\ [y, \varphi_{2m-n}](\infty) \end{pmatrix} = 0\}.$$

In the following, we will use (1.8), i.e., the relation between the θ_i's and the φ_i's, and Lemma 2.1 to prove the main result of this paper.

Let

$$B = ([\theta_i, \theta_j](0))^T_{1\leqslant i,j\leqslant 2m-n}. \tag{2.2}$$

Theorem 2.1 *Let $l(y)$ be a singular symmetric ordinary differential expression of order n with equal deficiency indices (m, m), where $[\frac{n+1}{2}] \leqslant m \leqslant n$, $\Pi(L_0) \cap \mathbb{R} \neq \emptyset$, and $\{\theta_1, \cdots, \theta_{2m-n}\}$ satisfy Theorem 1.1. Then, a linear subspace D in D_M is the domain of a self-adjoint extension of L_0 if and only if there exist complex matrices $M_{m\times n}$ and $N_{m\times(2m-n)}$ satisfying*

(1) $\operatorname{rank}(M \oplus N) = m$,

(2)$MQ^{-1}(0)M^* + NBN^* = 0$,

and such that

$$(3)\ D = \{y \in D_M : M \begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N \begin{pmatrix} [y, \theta_1](\infty) \\ \vdots \\ [y, \theta_{2m-n}](\infty) \end{pmatrix} = 0\}.$$

Proof *Necessity.* Let D be the domain of a self-adjoint extension of L_0. By Lemma 1.5, there exist $v_1, \cdots, v_m \in D_M$ satisfying the conditions (i), (ii) and (iii). From Theorem 1.1, each v_i can be uniquely written as

$$v_i = \tilde{y}_{0i} + \sum_{j=1}^{n} e_{ij} z_j + \sum_{j=1}^{2m-n} \tau_{ij}\theta_j, \tag{2.3}$$

where $\tilde{y}_{0i} \in D_0$. Substitute (1.8) into (2.3), we have

$$v_i = (\tilde{y}_{0i} + \sum_{k=1}^{2m-n} \tau_{ik} y_{0k}) + (\sum_{j=1}^{n} e_{ij} z_j + \sum_{k=1}^{2m-n} \sum_{j=1}^{n} \tau_{ik} d_{kj} z_j) + \sum_{k=1}^{2m-n} \sum_{j=1}^{2m-n} \tau_{ik} c_{kj} \varphi_j.$$

By Lemma 2.1, there exist matrices

$$M_1 = V^*(0)Q(0) \quad \text{and} \quad N_1 = (\bar{\tau}_{ij})_{m\times(2m-n)}(\bar{c}_{ij})_{(2m-n)\times(2m-n)}$$

satisfying the conditions $(1'), (2')$ and $(3')$, where $V(t)_{n\times m}$ denotes the Wronski matrix of $\{v_i(t);\ i = 1, \cdots, m\}$.

By (1.6), we have

$$\begin{aligned} B &= ([\theta_i, \theta_j](0))^T_{1\leqslant i,j\leqslant 2m-n} \\ &= ([\sum_{j=1}^{2m-n} c_{kj}\varphi_j, \sum_{j=1}^{2m-n} c_{lj}\varphi_j](\infty))^T_{1\leqslant k,l\leqslant 2m-n} \\ &= \bar{C}_1([\varphi_i, \varphi_j](\infty))^T_{1\leqslant i,j\leqslant 2m-n} C_1^T = \bar{C}_1 B_1 C_1^T, \end{aligned}$$

where $C_1 = (c_{ij})_{(2m-n)\times(2m-n)}$. By the proof of Theorem 1.1, we have $\operatorname{rank} C_1 = 2m - n$. Set

$$M = M_1 \quad \text{and} \quad N = N_1\bar{C}_1^{-1}. \tag{2.4}$$

Then,

$$\operatorname{rank}(M \oplus N) = \operatorname{rank}(M_1 \oplus N_1\bar{C}_1^{-1}) = \operatorname{rank}(M_1 \oplus N_1) = m,$$

$$\begin{aligned} MQ^{-1}(0)M^* + NBN^* &= M_1Q^{-1}(0)M_1^* + N_1\bar{C}_1^{-1}\bar{C}_1B_1C_1^T(\bar{C}_1^{-1})^*N_1^* \\ &= M_1Q^{-1}(0)M_1^* + N_1B_1N_1^* = 0, \end{aligned}$$

and for $y \in D_M$,

$$N\begin{pmatrix} [y,\theta_1](\infty) \\ \vdots \\ [y,\theta_{2m-n}](\infty) \end{pmatrix} = N\begin{pmatrix} [y, \sum_{j=1}^{2m-n} c_{1j}\varphi_j](\infty) \\ \vdots \\ [y, \sum_{j=1}^{2m-n} c_{2m-n,j}\varphi_j](\infty) \end{pmatrix} = N\bar{C}_1\begin{pmatrix} [y,\varphi_1](\infty) \\ \vdots \\ [y,\varphi_{2m-n}](\infty) \end{pmatrix}.$$

So,

$$M\begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N\begin{pmatrix} [y,\theta_1](\infty) \\ \vdots \\ [y,\theta_{2m-n}](\infty) \end{pmatrix} = M_1\begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N_1\begin{pmatrix} [y,\varphi_1](\infty) \\ \vdots \\ [y,\varphi_{2m-n}](\infty) \end{pmatrix}.$$

Therefore,

$$D = \{y \in D_M : M\begin{pmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} - N\begin{pmatrix} [y,\theta_1](\infty) \\ \vdots \\ [y,\theta_{2m-n}](\infty) \end{pmatrix} = 0\}.$$

Sufficiency. This follows by reversing the above arguments and then using Lemma 2.1. ■

Corollary 2.1 *Let M_1 and N_1 be the coefficient matrices for a self-adjoint domain with respect to $\varphi_1, \cdots, \varphi_{2m-n}$, and let M and N be the coefficient matrices for the domain with respect to $\theta_1, \cdots, \theta_{2m-n}$. Then, $M = M_1$ and $N = N_1\bar{C}_1^{-1}$.*

As mentioned in the introduction, the new characterization of the self-adjoint domains, given in Theorem 2.1, does not use any property of $l(y)$ at $t = +\infty$, since $Q(0)$ and B are defined at $t = 0$.

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