1 Permutations and combinations

We investigated the multiplication principle, and saw that it could be generalized to any finite sequence of tasks.

Result 1 If we have a sequence of \(k\) tasks, \(T_1, T_2, \ldots, T_k\), that can be done in \(n_1, n_2, \ldots, n_k\) ways, respectively, then the number of ways of completing the sequence of \(k\) tasks is \(n_1 \cdot n_2 \cdot \ldots \cdot n_k\).

This is a rather elementary idea - we illustrated it for \(k = 2, 3\) using both a tree model and a box model. In general, we shall use the box model for our discussion.

Example: Consider the set \(\{A, B, C, D\}\) - if we are going to make sequences of four letters from these three letters, then we could allow repetition or not.

- If we allow repetition (each letter can be used as many times as we wish), then we can view this as a series of three tasks involving three boxes.

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We have 4 ways of assigning the first letter (first task), 4 ways of assigning the second letter (task), and 4 for the final - this yields \(4 \cdot 4 \cdot 4 = 4^3 = 64\) such sequences.

- If we do NOT allow repetition, our box model is the same,

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but after we assign a letter to the first box in any of 4 ways, we are left with only 4 choices for the second box, and then 2 for the last box. This yields \(4 \cdot 3 \cdot 2 = 24\) such sequences.

In this chapter we see that this generalization of the multiplication principle can be used to count the number of possible sequences and sets of different types.

1.1 The number of elements in a Cartesian Product

We are familiar with the Cartesian product of two sets. They appeared in the study of relations and functions.

Example:

1. For a set \(S\), a binary relation is a subset of \(S \times S = \{(s_1, s_2) : s_1, s_2 \in S\}\). Equivalence relations and order relations on \(S\) were defined in terms of properties of the subsets.
   If \(S\) is a finite set with \(n\) elements, then \(S^2 = S \times S\) has \(n^2\) elements - we can count this set by realizing that the construction of an element, an ordered pair, consists of choosing a first coordinate, and then choosing a second coordinate - this is a sequence of two tasks which can each be done in \(n\) ways so there are \(n \cdot n = n^2\) elements of \(S^2\).

2. We defined a function \(f\) from the set \(S\) to the set \(T\) as a subset \(f\) of \(S \times T = \{(s, t) : s \in S, t \in T\}\) that had the property that for each \(s \in S\) there is exactly one \(t \in T\) such that \((s, t) \in f\).
   If \(S\) has \(n\) elements and \(T\) has \(m\) elements, then we can count the number of functions from \(S\) to \(T\). First we order the elements of \(S = \{s_1, s_2, \ldots, s_n\}\), and imagine that we have a sequence of \(n\) boxes.

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Any assignment of one element of \(T\) to each of the lower \(n\) boxes defines a function, and conversely, any function from \(S\) to \(T\) corresponds to such a distribution - place \(f(s_i)\) in the \(i^{\text{th}}\) box for each \(i\).

Now there are \(m\) ways to choose an entry for each of the \(n\) boxes, so there are \(m^n\) functions from \(S\) to \(T\).
Cartesian products are also familiar from our elementary studies of algebra and functions of a real variable - such as the line $y = 2x - 3$. Points in the real plane are labeled by 2 coordinates - $(x, y)$, where $x$ and $y$ are real numbers. Alternatively, if $\mathbb{R}$ is the set of all real numbers, then the real plane is coordinatized by $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Similarly, points in 3 dimensional space are given by 3 coordinates, $(x, y, z)$, and we could generalize our notation and write $\mathbb{R}^3$ for the set of all 3 element sequences of real numbers, or 3 dimensional coordinates.

**Definition 1** If we have $k$ sets, $S_1, S_2, \ldots, S_k$, then the Cartesian product of the sets is the set of all $n$-tuples that can be formed where the $i^{th}$ entry is from $S_i$, so $$S_1 \times S_2 \times \cdots \times S_k = \{(s_1, s_2, \ldots, s_k) : s_i \in S_i\}.$$  

If all $k$ sets are identical, then we write $S^k = S \times S \times \cdots \times S$ ($k$ copies of $S$).

If any of the sets in a Cartesian product is infinite, then the Cartesian product will have an infinite number of elements. However, if they are all finite sets, then we can easily enumerate the set using the multiplication principal.

**Result 2** If $S_1, S_2, \ldots, S_k$ are finite sets with $n_1, n_2, \ldots, n_k$ elements, respectively, then $S_1 \times S_2 \times \cdots \times S_k$ has $n_1 \cdot n_2 \cdots \cdot n_k$ elements.

*Comment:* We encounter elements of a Cartesian product in various ways. For instance, if $\mathcal{A}$ is the set of capital letters in the English alphabet, then $\mathcal{A}^3$ consists of triples, such as $(V, A, C)$, but we should realize that these elements also correspond to sequences of length 3, which we may write as $V, A, C$, or $VAC$ depending on context.

**Example:**

1. Suppose that a student chooses to label each of their computer files with a 2 symbol names, each consisting of a capital letter followed by a single digit.

   If $\mathcal{A}$ is the set of 26 letters, and $\mathcal{D}$ is the set of 10 digits, then the labels are pairs corresponding to elements of the Cartesian product $\mathcal{A} \times \mathcal{D}$, and there are $26 \cdot 10 = 260$ possible file names. (This problem is clearly an elementary application of the multiplication principle.)

2. If $\mathcal{A}$ is the standard alphabet of capital letters, a set with 26 elements, then $\mathcal{A}^5$ can be viewed as the set of all 5 letter words, meaningful or not, and there are $26^5$ such possible words. In this case, we would identify the 5-tuple $(B, R, A, N, D)$ with the word BRAND.

3. If $S = \{0, 1\}$, then $S^8$ is the number of 8-bit sequences (bytes), and there are $2^8 = 256$ different bytes. Notice this is a case where we write elements as sequences, rather than tuples - 01100011 and 00000100 are bytes. We do not remove leading 0's. Each element must have 8 entries.

The formula for the number of elements of a finite Cartesian product should not be necessary to memorize, it should become obvious as our familiarity with counting techniques develop.

The above examples are not difficult. However, the examination of a Cartesian product may allow us to enumerate an associated set. In the next example, we see that the number of subsets of a given finite set is easily related to binary sequences.

**Example: The number of subsets of a finite set**

For a set $S$, and $A \subseteq S$, we define the characteristic function\(^1\) of $A$ as $\chi_A : S \to \{0, 1\}$, by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

It should be clear that different subsets of $S$ will have different characteristic functions - they will differ whenever one considers an element of $S$ that is in one set, but not the other.

Note then that if $S = \{a_1, a_2, \ldots, a_k\}$, then we could use the elements of $S$ to label $k$ boxes,

\(^1\)The symbol $\chi$ is the Greek letter “chi”.
Now any assignment of 0’s and 1’s to these \( k \) boxes can be used to define a characteristic function of a subset \( A \). \( \chi_A(a_i) \) is the \( i \)th box’s entry, and \( a_i \in A \) if it is 1, otherwise it is not in \( A \).

All 1’s would be the subset \( S \), and all 0’s would be the emptyset, \( \emptyset \). The sequence \((1,1,0,0,\ldots,0)\) would correspond to the subset \( \{a_1,a_2\} \).

There are \( 2^k \) ways of assigning 0’s and 1’s to the boxes, so there are \( 2^k \) subsets of \( S \).

This example is actually related to the initial example where we enumerated the number of functions from a set \( S \) to a set \( T \), we found that if \( S \) had \( n \) elements and \( T \) had \( m \) then there were \( m^n \) functions from \( S \) to \( T \). In this situation, characteristic functions are from \( S \) to \( \{0,1\} \), so there are \( 2^n \) such functions.

The point is that we have associated each of the possible characteristic functions to a subset of \( S \).

A large set will have a large number of subsets, and it may be desirable to have a methodical way (algorithm) to enumerate or list all the subsets of \( S \). Upon consideration, we should realize that since every subset of \( S \) corresponds to one of the \( 2^k \) sequences of \( k \) 0’s and 1’s - that is, the elements of \( \{0,1\}^k \), once we have chosen an ordering of the elements of \( S \). So the problem of producing the set of all such subsets is equivalent to producing all length \( k \) sequences of 0’s and 1’s. We could do this by listing the binary equivalents of all the integers \( 0 \) through \( 2^{k+1} - 1 \), or by producing all such sequences using a binary tree. Of course there are other methods too.

1.2 Permutations

In this section we are concerned with forming sequences from a given set that do not have repeats.

**Definition 2** For a set \( S \) with \( n \) elements, a \( k \)-permutation of \( S \) is a sequence of \( k \) distinct elements from \( S \) (no repeats are allowed). If \( k = n \), then we just refer to an \( n \)-permutation of \( S \) as a permutation of \( S \).

Primarily, we remember that we are counting the number of possible sequences of \( k \) elements from \( S \) without repeats. This means that we would consider filling a line of \( k \) boxes, each with a different element of \( S \) - there are \( n \) choices for the first box, \( n - 1 \) for the second box, \( \ldots \), and \( n - (k - 1) \) for the last box.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>( k - 1 )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\ldots</td>
<td>( n )</td>
<td>( n - 1 )</td>
<td>( \ldots )</td>
<td>( (n - (k - 2)) )</td>
</tr>
</tbody>
</table>

So, if we denote the number of \( k \)-permutations of the \( n \) element set \( S \) by \( P(n,k) \), then

\[
P(n,k) = n(n - 1)\cdots(n - (k - 1)) = n(n - 1)\cdots(n - (k - 1)).
\]

We often use **factorial notation** in counting arguments.

**Definition 3** By convention \( 0! = 1 \), and for a positive integer \( n \)

\( n! = n(n - 1)\cdots3\cdot2\cdot1 \):

so, \( 1! = 1, 2! = 2\cdot1 = 2, 3! = 3\cdot2\cdot1 = 6 \),

and, in general \( (n + 1)! = (n + 1)\cdot n! \).

With this notation, we see that

**Result 3**

\[
P(n,k) = \frac{n(n - 1)\cdots(n - (k - 1))}{(n - k)!} = \frac{n(n - 1)\cdots(n - (k - 1))}{(n - k)!}
\]
Example: Let $S = \{A, B, C, \ldots, Z\}$ be the standard alphabet of 26 capital letters. The number of four letter words, without repeated letters, that can be formed is

$$P(26, 4) = \frac{26!}{(26 - 4)!} = \frac{26!}{22!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358,800.$$  

Of course, very few of these “words” correspond to actual words in our language.

Note that the number four letter words with repeats is $26^4$, so the number of four letter words with at least one repeated letter is $26^4 - P(26, 4)$.

The number of permutations of an $n$ element set $S$, that is the number of ways of ordering all of the elements on the set $S$, is

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = n!$$

So, for instance, although we are taught a unique way of listing the letters in the standard alphabet, there are $26!$ possible orderings - that’s $403,291,461,126,605,635,584,000,000$ possible orderings.

When we match elements from two different sets, a permutation model is often useful.

Example:

1. Consider a dancing lesson in which there are 12 men and 15 women participating. One may ask in how many ways can one form 12 dancing pairs consisting of 1 man and 1 woman. This is a permutation problem - imagine lining the 12 men against a wall, and then assigning a woman as a dance partner to each man - the ordering of the 12 men is analogous to our use of a sequence of 12 boxes (it provides the ordering), and there are $P(15, 12)$ of assigning the dancing partners.

2. In the same situation of assigning dancing partners, if there are 12 men and 12 women, then there will be 12! ways of assigning partners. Either the men or the women could be put in a line, and then there will be 12! ways of assigning partners.

Notice that in the first example, we used the smaller set (the men) to induce the ordering of the couples, and in the second, we could choose either set to induce an ordering, since they are both the same size.

Some problems involve several steps.

Example:

1. How many sequences of 4 different letters, followed by 3 different digits are there? This problem consists of two permutations, connected by the multiplication principle. Our first task, $T_1$ is to form the permutation of the letters - there are 26 letters, and there are $P(26, 4)$ such sequences, without repeats. Now our second task, $T_2$, is the formation of a 3-permutation of the 10 element set of digits (no repeats) - there are $P(10, 3)$ such sequences. By the multiplication principle, there are $P(26, 4) P(10, 3)$ such sequences.

2. Occasionly, there is a structural aspect to the problem considered. For instance, how many 5 digit sequences may be formed from the 10 digits, such that the the first four digits are different and the sum of the digits is congruent to 0, modulo 10. To solve this, we realize that the first four digits are a permutation, and the fifth digit is uniquely determined by the previous 4 digits - we do NOT get to choose the last digit, it is uniquely determined by the first 4 - so there are $P(10, 4)$ such sequences: for instance, 2,4,5,2 determines the last digit to be 7, while the sequence 2,3,0,5 determines a final digit of 0.

Occasionally, a problem has restrictions that require some modification on how we interprete our sequence of tasks.
Example: In how many ways can one form sequences of 6 different digits, in which the 1st, 3rd and 5th digits are odd, and the 2nd, 4th and 6th can be any digits. In such problems, where there are restrictions on some of the elements of the sequences, we assign the restricted portions first. The restrictions are on the 3 odd numbered entries - there are \( P(5, 3) \) ways to assign the odds to the 1st, 3rd, and 5th entries. Once we have made an assignment, there are 7 digits remaining to permute in the remaining 3 boxes - there are \( P(7, 3) \) ways to assign the even entries on the sequence. This yields a total of \( P(5, 3) \cdot P(7, 3) = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(7 \cdot 6 \cdot 5) = 25,200 \) ways to form such sequences.

1.3 Permutations and combinations

In the previous material we discussed permutations: given an \( n \) element set \( S \), we found

- the number of \( k \)-permutations, \( k \) element sequences of distinct elements of \( S \) was
  \[
P(n, k) = \frac{n!}{(n-k)!}
  \]
- and, in the case where we permute all the elements of the set, a permutation of \( S \), then there are \( n! \) such permutations.

Now, for a set \( S \), a \( k \)-combination of \( S \) is a subset of \( k \) (distinct) elements of \( S \). Notice that if \( S \) has \( n \) elements, when there is 1 \( n \)-combination of \( S \) - the whole set. The number of 0-combinations of \( S \) is also 1 - the only subset that can be formed is the emptyset.

Our formula for the number of \( k \)-combinations of an \( n \) element set is derived from our formula for the number of \( k \)-permutations of the set.

Let \( S \) be an \( n \) element set, and let \( C(n, k) \) denote the number of \( k \)-combinations of \( S \). Now,

- each \( k \)-combination is a distinct subset of \( k \) elements from \( S \), so if we have such a set, we can then form \( k! \) permutations of this subset. Each of these permutations will be a different ordering of the \( k \) elements.
- this is true for each of the possible \( k \)-combinations of \( S \), and no two different \( k \)-combinations will produce the same \( k \)-permutation since there is at least one different element in the two combinations.
- Then, we see that \( (k!) \cdot C(n, k) = P(n, k) \), since every \( k \)-permutation of \( S \) is a permutation of a \( k \)-combination of \( S \).

Theorem 1 If \( S \) is a set with \( n \) elements, then the number of \( k \)-combinations of \( S \) satisfies the relationship

\[
(k!) \cdot C(n, k) = P(n, k),
\]

or

\[
C(n, k) = \frac{n!}{k!(n-k)!}.
\]

Example: Consider \( S = \{A, B, C\} \).

1. the number of 0-combinations of \( S \) is \( C(3, 0) = \frac{3!}{0!(3-0)!} = \frac{3!}{1 \cdot 3!} = \frac{3!}{3!} = 1 \). The 0-combination is \( \emptyset \).
2. the number of 1-combinations of \( S \) is \( C(3, 1) = \frac{3!}{1!(3-1)!} = \frac{3!}{1! \cdot 2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3 \).
   The 1-combinations are \( \{A\}, \{B\}, \{C\} \).
3. the number of 2-combinations is \( C(3, 2) = 3 \) also, and they are \( \{A, B\}, \{A, C\}, \{B, C\} \).
4. and, as noted above, the only 3-combination of \( S \) is the whole set. This agrees with our counting argument \( C(3, 3) = \frac{3!}{0!3!} = \frac{6}{6} = 1 \).
A fundamental distinction between applications involving permutations and combinations is the implied ordering (or differences) in the choice of objects.

**Example:** Consider a club with 30 students.

1. If we are to choose a president, vice president and secretary, then these positions are different - so this is a permutation, there are \( P(30, 3) \) ways of choosing the club officers.
2. On the other hand, if we are to select a 3 member governance body for the club, where each of the members have equal responsibilities, then we are forming a subset of 3 members of the club - this is a combination. There are \( C(30, 3) \) ways of choosing this governance body.

We used the box model to count possibilities when considering permutations. It is also helpful when considering problems that can be interpreted as combinations.

**Example:** How many 8-bit strings (strings of 0’s and 1’s) can be formed that have exactly 5 0’s in them?

At first, this does not seem like a combination problem, but it can be viewed in a manner that makes it a combination problem.

When we form an 8-bit sequence we can imagine that we are filling 8 boxes with a 0 or a 1.

\[
\begin{array}{cccccccc}
\scalebox{0.5}{0} & \scalebox{0.5}{0} & \scalebox{0.5}{0} & \scalebox{0.5}{0} & \scalebox{0.5}{1} & \scalebox{0.5}{1} & \scalebox{0.5}{1} & \scalebox{0.5}{1}
\end{array}
\]

To solve our problem, note that we just must pick 5 of the 8 boxes to put a 0 into, and then we will place a 1 in all the remaining boxes.

There are \( C(8, 5) \) ways of choosing the 5 boxes to put a 0 into, so there are

\[
C(8, 5) = \frac{8!}{5!(8-5)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56
\]

such sequences.

### 1.4 Multisets and permutations of things not all different.

A classic problem in combinatorics is the question *in how different ways can the letters in MISSISSIPPI be reordered*. This is clearly a permutation problem: there are 11 letters, but there are fewer than 11! rearrangements of the word. In particular, for any one arrangement, there are 4! ways of rearranging the S’s that result in the identical word.

This is an example of a permutation of a multiset, or of a sequence of things not all different. The last example of the previous section - where we enumerated the number of bytes with exactly five 0’s and three 1’s, is an example of such a problem - each byte enumerated is an ordering of five 0’s and three 1’s. Our method of solution to that problem is pertinent to our approach to this problem.

**Example:** Consider the word MISSISSIPPI - it has one M, two P’s, four I’s, and four S’s.

We can view a permutation of this as a process of filling a sequence of 11 boxes. A permutation of such a word is determined by the position of the letters, not the order in which we place them.

To assign the S’s we choose 4 boxes to put them into, this can be done in \( C(11, 4) \) ways, leaving 7 empty boxes. Now we can assign the 4 I’s in \( C(7, 4) \) ways, leaving 3 boxes. Then we choose places for the P’s in \( C(3, 2) \) ways, and finally, that leaves \( C(1, 1) \) to assign the single M. This was a sequence of 4 tasks, and we employ the multiplication principal, to see that there are

\[
C(11, 4) \cdot C(7, 4) \cdot C(3, 2) \cdot C(1, 1) = \frac{11!}{4!7!} \cdot \frac{7!}{4!3!} \cdot \frac{3!}{2!1!} \cdot \frac{1!}{1!1!} = \frac{11! \cdot 7! \cdot 3! \cdot 1!}{4! \cdot 4! \cdot 2! \cdot 1!} = \frac{11!}{4!2!}.
\]

This example illustrates the basis of the methodology that we will discuss, but it is helpful to introduce some terminology. Remember that a set cannot contain duplicates of the same element, so when we considered permuting the letters of MISSISSIPPI we were not considering the permutation of the letters in a set. The set of letters used in the word is \{M, I, S, P\}. With this in mind we define a multi-set
Definition 4 A multi-set is a collection of objects with repeats allowed. If the objects are $A_1, A_2, \ldots, A_k$, and there are $n_1, n_2, \ldots, n_k$ copies of each object, respectively, then we denote the collection \{n_1 \cdot A_1, n_2 \cdot A_2, \ldots, n_k \cdot A_k\}, and sometimes say that we have $n_1$ objects of type $A_1$, etc..

For instance, the letters in MISSISSIPPI form the multiset \{1 \cdot M, 2 \cdot P, 4 \cdot I, 4 \cdot S\}.

The formula for the number of distinct permutations of a multiset generalizes the pattern that we saw in our preliminary investigation of the rearrangements of the word MISSISSIPPI, and the proof will follow from a generalization of the counting argument that we used in that example.

Result 4 The number of distinct permutations of the multiset \{n_1 \cdot A_1, n_2 \cdot A_2, \ldots, n_k \cdot A_k\} is denoted $${n \choose n_1, n_2, \ldots, n_k}$$ where $n = n_1 + n_2 + \cdots + n_k$, and

$${n \choose n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Outline of proof: For the multi-set \{n_1 \cdot A_1, n_2 \cdot A_2, \ldots, n_k \cdot A_k\}, where $n = n_1 + n_2 + \cdots + n_k$, we imagine that we have $n$ boxes. We first distribute the $n_1$ copies of $A_1$ in a choice of $n_1$ of the $n$ boxes, there are $C(n, n_1)$ ways of choosing these boxes, and we are left with $n - n_1$ open boxes.

We then distribute the $n_2$ copies of $A_2$ into a choice of $n - n_1$ boxes - there are $C(n - n_1, n_2)$ ways of choosing these boxes, and we will be left with $n - n_1 - n_2$ open boxes.

As above, we continue this process and apply the multiplication principle:

$${n \choose n_1, n_2, \ldots, n_k} = C(n, n_1) \cdot C(n - n_1, n_2) \cdot \cdots \cdot C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k).$$

The pattern of cancellation observed in the first example generalizes, and we obtain

$${n \choose n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Example:

1. The number of distinct arrangements of the multiset \{3 \cdot A, 8 \cdot B, 12 \cdot C\} is

$$\left( \begin{array}{c} 23 \\ 3, 8, 12 \end{array} \right) = \frac{23!}{3! 8! 12!}.$$  

2. The number of ways of rearranging the letters in CALCULUS is the number of permutations of the multiset \{1 \cdot A, 2 \cdot C, 2 \cdot L, 2 \cdot U, 1 \cdot S\}, or

$$\left( \begin{array}{c} 8 \\ 1, 2, 2, 2, 1 \end{array} \right) = \frac{8!}{1! 2! 2! 1!} = \frac{8!}{2! 2! 2! 1!}.$$  

3. Consider walking on a standard block patterned urban downtown. If you have to walk 8 blocks North and 5 blocks West, then in how many ways can you choose a route. A route can be seen to correspond to a permutation of the multi-set \{8 \cdot N, 5 \cdot W\}. For instance, NNNNNNNNNWWWWW would correspond to a walk 8 blocks North and 5 blocks West. The number of possible routes is

$$\left( \begin{array}{c} 13 \\ 8, 5 \end{array} \right) = \frac{13!}{8! 5!}.$$  

1.5 Combinations involving multisets

We previously discussed the problem of permuting a multiset of “things not all different”.

Consider two similar problem concerning “things not all different”.

1. Suppose that we have $n$ indistinguishable objects, such as identical tennis balls.

In how many ways could we distribute the $n$ balls to $r$ different players? (It doesn’t matter which tennis balls a player gets, just how many.)
2. Given an unlimited number of duplicates of a collection of objects (or *types* of objects as above), in how many ways can we form a *multi-set* of *n* total objects?

We shall see that both of these questions are directly related to the investigation of linear equations of he form

\[ x_1 + x_2 + \cdots + x_r = n \]

where \( x_i \geq 0 \).

1. If we have 10 indistinguishable tennis balls, and 3 distinguishable players, then a distribution of the 10 balls among the three players (# 1, # 2, # 3) corresponds to a solution to the equation

\[ x_1 + x_2 + x_3 = 10 \]

where each of \( x_1, x_2 \) and \( x_3 \) are non-negative integers. For instance, the solution \( x_1 = 2, x_2 = 0 \) and \( x_3 = 8 \) corresponds to the distribution of 2 balls to player # 1, 0 to # 2 and 8 to # 3.

2. Alternately, suppose that we have an unlimited number of each of the objects of type \( A_1, A_2 \) and \( A_3 \) and that we wish to form a multiset of size 10 of these types of objects. Any such multiset corresponds to a solution to the equation

\[ x_1 + x_2 + x_3 = 10 \]

where each of \( x_1, x_2 \) and \( x_3 \) are non-negative integers. In this case, the solution \( x_1 = 2, x_2 = 0 \) and \( x_3 = 8 \) corresponds to the multiset \( \{2 \cdot A_1, 0 \cdot A_2, 8 \cdot A_3\} \) of size 10.

This is our motivation for studying the question of the number of integer solutions to linear equations of the form

\[ x_1 + x_2 + \cdots + x_r = n \]

have where each \( x_i \geq 0 \).

Equations of this type are encountered in other contexts, and we investigate the general question of counting the number of solutions to such equations first - our enumerations, as above, are applications of this general question. The particular case,

\[ x_1 + x_2 + \cdots + x_r = n \]

have where each \( x_i \geq 1 \) is discussed first - the *lower limits are 1 here*. It is particularly easy to model, and has a pleasing visual aspect.

**The number of solutions to an equation.**

We examine the specific question *How many integer solutions does*

\[ x_1 + x_2 + \cdots + x_r = n \]

*have where each \( x_i \geq 1 \)?*

There is a rather simple model for this problem - called the *stick model* - we line up \( n \) sticks and divide them into \( r \) nonempty ordered piles to represent a solution - notice that it only takes \((r - 1)\) dividers to form \( r \) subcollections of the line of sticks. Also note that we will have at most one divider between any two sticks.

**Example:** Consider the equation \( x_1 + x_2 = 8 \), where \( x_1, x_2 \geq 1 \).

A solution corresponds to a division of the string of 8 sticks

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ | & | & | & | & | & |
\hline
\end{array}
\]

into two parts - this takes one divider.

The solution \( 3 + 5 = 8 \) could be represented by

\[
\begin{array}{|c|c|c|}
\hline
\ | & | & |
\hline
1 & 2 & 3
\end{array}
\begin{array}{|c|c|c|c|c|c|}
\hline
\ | & | & | & | & | |
\hline
1 & 2 & 3 & 4 & 5
\end{array}
\]
Now from this we should see that if we place the arrow between any two sticks, then we’ll get a different solution to the equation. There are 7 spaces in between the 8 sticks, so we can choose any one of 7 places for the division of the line of sticks - so there are 7 solutions. They are (1,7), (2,6), (3,5), (4,4), (5,3), (6,2) and (7,1).

This pattern continues - if we wish to find the number of solutions to
\[ x_1 + x_2 + \cdots + x_r = n \]
have where each \( x_i \geq 1 \), we would consider a line of \( n \) sticks, and then notice that we need only make \( (r-1) \) separations of this line, to form \( r \) subcollections of the line of sticks, and there are \((n-1)\) places to make the divisions.

So, there are \( C(n-1, r-1) \) solutions to the equation
\[ x_1 + x_2 + \cdots + x_r = n \]
have where each \( x_i \geq 1 \).

**Example:** Suppose we have \( x_1 + x_2 + x_3 + x_4 = 16 \), then any solution where each \( x_i \geq 1 \), corresponds to a division of the line of 16 sticks
\[
| | | | | | | | | | | | | | | |
\]
such as
\[
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 1 \quad 1 \quad 2 \quad 3
\]
Which corresponds to the solution \( x_1 = 2, x_2 = 10, x_3 = 1 \) and \( x_4 = 3 \).

This is a model for a situation similar to our introductory question about distributing tennis balls. Consider the question *In how many ways can 16 tennis balls be distributed to four players, in such a way each player gets at least one tennis ball?*

We order/enumerate the players - 1, 2, 3, 4. And for any distribution of the balls, let \( x_1, x_2, x_3 \) and \( x_4 \) be the number of balls each player receives, respectively. Then any proper distribution will satisfy the equality \( x_1 + x_2 + x_3 + x_4 = 16 \), with \( x_i \geq 1 \) for each \( i \).

We have a total of \( n = 16 \) indistinguishable objects, distributed to \( k = 4 \) distinguishable players. The number of distinct ways of doing this is
\[
C(n-1, k-1) = C(15, 3) = \frac{15!}{3!12!} = \frac{15 \cdot 14 \cdot 13 \cdot 12!}{3!12!} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 5 \cdot 13 = 405
\]
distinct ways of distributing the tennis balls.

**The more general question**

When we consider a linear equation
\[ x_1 + x_2 + \cdots + x_r = n \]
our problem may require different different bounds than each \( x_i \geq 1 \). To handle such situations, we use substitution.

**Example:** How many integer solutions does
\[ x_1 + x_2 + x_3 = 10 \]
have such that \( x_1 \geq 0, x_2 \geq 3 \) and \( x_3 \geq -2? \)

We add/subtract to put our inequalities in the form \( x_i - a \geq 1 \), and introduce new variables \( y_i \)
\[ y_1 = x_1 + 1 \geq 1, \quad y_2 = x_2 - 2 \geq 1 \quad \text{and} \quad y_3 = x_3 + 3 \geq 1, \]
and we substitute \( x_1 = y_1 - 1, \quad x_2 = y_2 + 2, \quad x_3 = y_3 - 3 \), in our equation to get a related equality: the number of solutions to
\[ x_1 + x_2 + x_3 = 10 \]
such that $x_1 \geq 0$, $x_2 \geq 3$ and $x_3 \geq -2$, is the same as the number of solutions to

$$(y_1 - 1) + (y_2 + 2) + (y_3 - 3) = 10$$

or

$$y_1 + y_2 + y_3 = 12$$

such that $y_i \geq 1$ for each $i$. There are $C(10 - 1, 3 - 1) = C(9, 2) = 36$ solutions to this equation, and the original one.

For instance, the solution $y_1 = 6$, $y_2 = 4$, $y_3 = 2$, corresponds to the solution, $x_1 = 6 - 1 = 5$, $x_2 = 4 + 2 = 6$ and $x_3 = 2 - 3 = -1$ of the original equation.

The special case of counting the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_r = n$$

where $x_i \geq 0$ deserves special attention and has a memorable interpretation when using a stick model.

Notice that each inequality $x_i \geq 0$, yields $y_i = x_i + 1 \geq 1$, and $x_i = y_i - 1$.

Using the substitution method as above we get the equation

$$y_1 + y_2 + \cdots + y_r = n + r$$

where $y_i \geq 1$ for each $i$. The number of solutions to this equation, and thus our original equation in $x_i$’s, has $C(n + r - 1, r - 1)$ solutions.

Example: Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

such that $x_1, x_2, x_3, x_4 \geq 0$. We model a solution for this by forming a line of $19 = 15 + 4$ sticks - 15 for the total, and one more for each variable, and then dividing the line into 4 segments, by choosing 3 places to break this line - one for each + sign.

| | | | | | | | | | | | | | | | | | |

such as

| | | | | | | | | | | | | | | | | | |

↑ ↑

Now, when we interprete this as a solution to the system, we start counting each segment or pile at 0 - the minimal value the variable may have.

| | | | | | | | | | | | | | | | | | |

0 1 0 1 2 3 4 5 6 7 8 9 0 0 1 2 3 4 5

Which yields the solution - $x_1 = 1$, $x_2 = 9$, $x_3 = 0$ and $x_4 = 5$.

Result 5 The number of solutions in integers to

$$x_1 + x_2 + \cdots + x_r = n$$

where $x_i \geq 0$ for each $i$ is $C(n + r - 1, r - 1)$ solutions.