Math 232 Fall 2002

Write $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{0} = \langle 0, 0, 0 \rangle$. Define $\mathbf{v} \pm \mathbf{w} = \langle v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3 \rangle$ and $t\mathbf{v} = \langle tv_1, tv_2, tv_3 \rangle$. The dot product is $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$.

The length is $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. With $t = 1/|\mathbf{v}|$, $\mathbf{u} = t\mathbf{v}$ has length 1.

Let $\mathbf{v}$ and $\mathbf{w}$ span two sides of a triangle. Represent the third side by $\mathbf{v} \times \mathbf{w}$. The angle between $\mathbf{v}$ and $\mathbf{w}$ be given by $\theta$. According to the Law of Cosines, it must be that

$$|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{w}|^2 + |\mathbf{v}|^2 - 2|\mathbf{v}||\mathbf{w}||\cos \theta|.$$

In terms of the angle $\theta$ between $\mathbf{v}$ and $\mathbf{w}$ this is $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}||\sin \theta|.$

**Application:**

Given two points $P$ and $Q$ let the vector $\mathbf{p}$ and the vector $\mathbf{q}$ go from the origin to $P$ and $Q$ respectively. Let $\mathbf{v} = \mathbf{q} - \mathbf{p}$. The parametric curve $\mathbf{r}'(t) = \mathbf{p} + t\mathbf{v}$ traces a straight line, which passes through $P$ when $t = 0$ and $Q$ when $t = 1$. The midpoint of the straight line segment between $P$ and $Q$ is $\mathbf{r}'(\frac{1}{2}) = \mathbf{p} + \frac{1}{2}\mathbf{v} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$. If there is a mass $m_p$ at $P$ and a mass $m_Q$ at $Q$, then the center of mass is $m_p \mathbf{p} + m_Q \mathbf{q}$.

**Application:**

Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$. Connect $PQ$ so that $\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$.

The distance between $P$ and $Q$ is given by

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}.$$ 

All points $(x, y, z)$ a distance $r > 0$ from $P$ satisfy $(x - p_1)^2 + (y - p_2)^2 + (z - p_3)^2 = r^2$. This is the equation of a sphere of radius $r$ with center at $P$.

**Application:**

Let $\theta$ be the angle between $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. It must be that

$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right).$$

In particular, $\theta = \pi / 2$ if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. The vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

**Application:**

Let $P = (p_1, p_2, p_3)$ be some given point in a plane. Let $Q = (x, y, z)$ be an arbitrary point in the same plane. The vectors $\mathbf{v} = \langle x - p_1, y - p_2, z - p_3 \rangle$ are always in the plane. Let $\mathbf{n} = \mathbf{0}$ be a fixed vector such that $\mathbf{n} \cdot \mathbf{v} = 0$. Any such $\mathbf{n}$ is a normal vector to the plane. If $\mathbf{n} = \langle A, B, C \rangle$, then $\mathbf{n} \cdot \mathbf{v} = 0$ is equivalent to $Ax + By + Cz = Ap_1 + Bp_2 + Cp_3 = D$.

**Application:**

Let $\theta$ be the angle between $\mathbf{v}$ and $\mathbf{w}$. The quantity $|\mathbf{w}| \cos \theta$ is the size of the adjacent side, in
the direction of $\mathbf{V}$, of a triangle with hypotenuse along $\mathbf{W}$. This value is $\mathbf{U} \cdot \mathbf{W}$ where $\mathbf{U}$ is a unit vector in the direction given by $\mathbf{V}$. The vector $(\mathbf{U} \cdot \mathbf{W})\mathbf{U}$ is the projection of $\mathbf{W}$ onto $\mathbf{V}$.

**Application:**
Let $\mathbf{n} = \langle A, B, C \rangle$ be a normal vector to a plane $Ax + By + Cz = D$ with $\mathbf{p}$ a vector pointing to a point $(p_1, p_2, p_3)$ in the plane. Let $\mathbf{U}$ be the unit vector in the direction given by $\mathbf{n}$. The triangle created by $\mathbf{p}$ as the hypotenuse and $\mathbf{U}$ along one of the short sides has $(\mathbf{p} \cdot \mathbf{U})\mathbf{U}$ pointing to a point in the plane. All other points in the plane must be further away from the origin. It follows that the distance from the plane to the origin is $|\mathbf{p} \cdot \mathbf{U}| = D / \sqrt{A^2 + B^2 + C^2}$. The distance from the plane to a point $(q_1, q_2, q_3)$ pointed to by $\mathbf{q}$ is the same formula with $\mathbf{p} - \mathbf{q}$ replacing $\mathbf{p}$ so $D = A(p_1 - q_1) + B(p_2 - q_2) + C(p_3 - q_3)$.

**Application:**
Since $\mathbf{V} = t\mathbf{W}$ if and only if $\mathbf{V} \times \mathbf{W} = \mathbf{0}$, it follows that two vectors are parallel if and only if their cross product vanishes.

**Application:**
The area of the parallelogram spanned by $\mathbf{V}$ and $\mathbf{W}$ is given by $|\mathbf{V} \times \mathbf{W}|$.

**Application:**
The area of a triangle with vertices $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, and $R = (r_1, r_2, r_3)$ is $|\mathbf{V} \times \mathbf{W}|/2$, where $\mathbf{V} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$ and $\mathbf{W} = \langle r_1 - p_1, r_2 - p_2, r_3 - p_3 \rangle$ represents two of its sides.

**Application:**
The parallelepiped spanned by $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ has volume $|\mathbf{U} \cdot (\mathbf{V} \times \mathbf{W})|$

**Application:**
Visualize the vectors perpendicular to the line $\mathbf{p} + t\mathbf{V}$. Consider the opposite side in a right angle triangle with hypotenuse $\mathbf{p}$ and adjacent side $t\mathbf{V}$ for some $t$. With $\theta$ the angle between $\mathbf{p}$ and $\mathbf{V}$, the length of the opposite side is $|\mathbf{p}| \sin \theta$. This value is equal to $|\mathbf{U} \times \mathbf{p}|$ where $\mathbf{U}$ is a unit vector in the direction given by $\mathbf{V}$. It follows that the distance from the line $\mathbf{p} + t\mathbf{V}$ to the origin is $|\mathbf{U} \times \mathbf{p}|$. The distance from the line to some arbitrary point $\mathbf{q}$ is $|\mathbf{U} \times (\mathbf{p} - \mathbf{q})|$

**Application:**
Consider the two lines $\mathbf{p} + t\mathbf{V}$ and $\mathbf{q} + s\mathbf{W}$. Visualize the triangle with hypotenuse $\mathbf{p} - \mathbf{q}$ and adjacent side the perpendicular segment between the two lines. If the lines are parallel, then the distance between the two lines equals the distance from $\mathbf{q}$ to the first line. If the lines are not parallel, then the vector $\mathbf{V} \times \mathbf{W} = \mathbf{U}$ is perpendicular to both lines. Let $\theta$ be the angle between $\mathbf{p} - \mathbf{q}$ and $\mathbf{V} \times \mathbf{W}$. The length of the opposite side of the triangle is $|\mathbf{p} - \mathbf{q}| \sin \theta$. Let $\mathbf{U}$ be a unit vector in the direction given by $\mathbf{V} \times \mathbf{W}$. The distance between the two lines is $|\mathbf{U} \times (\mathbf{p} - \mathbf{q})|$

**Application:**
Suppose a line of direction $\mathbf{V}$ bounces of a surface with normal $\mathbf{n}$. The reflected line’s direction $\mathbf{W}$ is given by $\mathbf{W} = \mathbf{V} - 2(\mathbf{V} \cdot \mathbf{n})\mathbf{n}$.