5 The general linear optimization problem.

5.1 Primal and dual problem.

Let \( n, m \in \mathbb{N} \). Assume that \( x, c \in \mathbb{R}^n \) and \( \lambda, b \in \mathbb{R}^m \). Let \( A \) be an \( m \times n \) real matrix. The primal problem is given by

\[
\max c^T x \quad \text{when} \quad \begin{cases} Ax \leq b \\ x \geq 0 \end{cases}
\]

The corresponding dual problem is given by

\[
\min \lambda^T b \quad \text{when} \quad \begin{cases} A^T \lambda \geq c \\ \lambda \geq 0 \end{cases}
\]

5.2 Verifiable solutions.

Whenever a ‘solution’ is given for an optimization problem it is prudent to be suspicious. In the case of linear problems there is a wonderful optimality test available. To apply the test both the primal problem and the corresponding dual problem must be solved. General-purpose programs such as Solver in Excel capably handle both problems. To verify a solution-pair one must check one equality and \( 2(n + m) \) inequalities. The test is based on the following result.

**VERIFICATION THEOREM**

If \( \hat{x} \in \mathbb{R}^n \) is feasible in the primal problem and \( \hat{\lambda} \in \mathbb{R}^m \) is feasible in the dual problem, then \( c^T \hat{x} = \hat{\lambda}^T b \) implies that both \( \hat{x} \) and \( \hat{\lambda} \) are optimal.

**Proof:**

Suppose \( x \in \mathbb{R}^n \) is any feasible point in the dual problem, then

\[
c^T x \leq (A^T \hat{\lambda})^T x = (\hat{\lambda}^T (A^T)^T) x = (\hat{\lambda}^T A) x = \hat{\lambda}^T (A x) \leq \hat{\lambda}^T b = c^T \hat{x},
\]

and hence \( x \in \mathbb{R}^n \) is optimal in the primal problem.

Suppose \( \lambda \in \mathbb{R}^m \) is any feasible point in the dual problem, then

\[
\lambda^T b \geq \lambda^T (A \hat{x}) = (\lambda^T A) \hat{x} = (A^T \lambda)^T \hat{x} \geq c^T \hat{x} = \hat{\lambda}^T b,
\]

and hence \( \hat{\lambda} \in \mathbb{R}^m \) is optimal in the dual problem.

**Exercise**

Examine the proof of the Verification Theorem. Pinpoint where in the proof the fact that \( \hat{x} \) satisfies \( \hat{x} \geq 0 \) is used. Pinpoint where in the proof the fact that \( \hat{\lambda} \) satisfies \( \hat{\lambda} \geq 0 \) is used.
5.3 Equal objectives.

Suppose $x \in \mathbb{R}^n$ is feasible in the primal problem and $\lambda \in \mathbb{R}^m$ is feasible in the dual problem. The dual objective always exceeds the primal objective since
\[
\mathbf{c}^T \mathbf{x} \leq (\mathbf{A}^T \mathbf{\lambda})^T \mathbf{x} = (\mathbf{\lambda}^T (\mathbf{A}^T))^T \mathbf{x} = (\mathbf{\lambda}^T \mathbf{A})^T \mathbf{x} = \mathbf{\lambda}^T (\mathbf{A} \mathbf{x}) \leq \mathbf{\lambda}^T \mathbf{b}.
\]

If $\hat{\lambda} \in \mathbb{R}^m$ is optimal in the dual problem, then $\mathbf{c}^T \mathbf{x} \leq \mathbf{\lambda}^T \mathbf{b}$ for all feasible points $\mathbf{x} \in \mathbb{R}^n$ of the primal problem. If $\hat{\mathbf{x}} \in \mathbb{R}^n$ is optimal in the primal problem, then $\mathbf{c}^T \mathbf{x} \leq \mathbf{\lambda}^T \mathbf{b}$. Observe that the Verification Theorem is not applicable if $\mathbf{c}^T \mathbf{x} < \mathbf{\lambda}^T \mathbf{b}$. It is therefore very exciting to announce that $\mathbf{c}^T \mathbf{x} < \mathbf{\lambda}^T \mathbf{b}$ never happens at optimal points. This fact is one of the consequences of the so-called Fundamental Theorem of linear optimization. The Fundamental Theorem is an example of a ‘deeper’ mathematical result, and its details are deferred for now.

5.4 Complementary Slackness.

**COMPLEMENTARY SLACKNESS THEOREM**

Assume $\mathbf{x} \in \mathbb{R}^n$ is feasible in the primal problem and $\hat{\lambda} \in \mathbb{R}^m$ is feasible in the dual problem. Let $\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{x} \in \mathbb{R}^m$ be the slack in the primal problem and $\hat{\mu} = \mathbf{A}^T \hat{\lambda} - \mathbf{c} \in \mathbb{R}^n$ the slack in the dual problem. Suppose that $\mathbf{c}^T \mathbf{x} = \mathbf{\lambda}^T \mathbf{b}$ so that $\hat{\mathbf{x}}$ and $\hat{\mathbf{\lambda}}$ are optimal, then
\[
\begin{align*}
(1) \quad & y_j > 0 \Rightarrow \hat{\lambda}_j = 0, \quad j \in \{1, \ldots, m\} \\
(2) \quad & \hat{\mu}_i > 0 \Rightarrow \hat{x}_i = 0, \quad i \in \{1, \ldots, n\} \\
(3) \quad & \hat{x}_i > 0 \Rightarrow \hat{\mu}_i = 0, \quad i \in \{1, \ldots, n\} \\
(4) \quad & \hat{\lambda}_j > 0 \Rightarrow y_j = 0, \quad j \in \{1, \ldots, m\}
\end{align*}
\]

**Proof:**

First observe that
\[
(\mathbf{A}^T \hat{\lambda} - \mathbf{c})^T \mathbf{x} + \mathbf{\lambda}^T (\mathbf{b} - \mathbf{A}\mathbf{x}) = 0,
\]
and
\[
(\mathbf{A}^T \hat{\lambda} - \mathbf{c})^T \mathbf{x} + \mathbf{\lambda}^T (\mathbf{b} - \mathbf{A}\mathbf{x}) = \hat{\mu}^T \mathbf{x} + \mathbf{\lambda}^T \mathbf{s},
\]
so
\[
\hat{\mu}^T \mathbf{x} + \mathbf{\lambda}^T \mathbf{s} = 0.
\]

Every single term in both of the matrix products is non-negative and therefore equal to zero. Each term is a product of two factors. If one factor is greater than zero, then the other factor must be equal to zero.

**Example**

The primal problem
\[
\text{max } x_1 - 2x_2 + 3x_3 \quad \text{when} \quad \begin{cases} 4x_1 + 5x_2 + 6x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{cases}
\]
has dual

$$\text{min } 7\lambda \text{ when } \begin{cases} 4\lambda \geq 1 \\ 5\lambda \geq -2 \\ 6\lambda \geq 3 \\ \lambda \geq 0 \end{cases}$$

The solution of the dual problem is given by $\hat{\lambda} = \frac{1}{7}$. The slack is given by

$$\hat{\mu} = \begin{bmatrix} 1 \\ \frac{9}{7} \\ 0 \end{bmatrix}.$$ 

Using complementary slackness it follows that $x_1 = 0$ and $x_2 = 0$. It also follows that $y_1 = 0$ and $4x_1 + 5x_2 + 6x_3 = 7$, and hence $x_3 = \frac{7}{6}$.

### 5.5 Conversion.

Suppose $\bar{x} \in \mathbb{R}^n$ is optimal with $\bar{y} \in \mathbb{R}^m$ the corresponding slack. If the augmented point is a basic **non-degenerate** solution so that

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

is such that exactly $n$ entries are zero and all other entries are positive, then the dual solution is found from the linear system $A^T \hat{\lambda} - \hat{\mu} = c$. This system has $n + m$ variables and $n$ equations. By the complementary slackness $m$ of the variables are equal to zero. The remaining $n$ variables are given as a solution to the system.