Let \( f(x) = x^3 - 3x + 6 \). (p 199 #9)

(a) [6 pts] Find \( f'(x) \) and the critical points of \( f(x) \).

\[
 f'(x) = 3x^2 - 3 \quad \text{Set } f'(x) = 0 \text{ to get } 3x^2 - 3 = 0 \quad 3x^2 = 3 \quad x^2 = 1 \quad x = \pm 1
\]

(b) [4 pts] Find the intervals on which \( f(x) \) is increasing and decreasing.

The critical points separate the number line into the intervals \((\infty, -1)\), \((-1, 1)\), and \((1, \infty)\). We need to check the sign of the first derivative at one point in each of the intervals. You could choose \(-2\), \(0\), and \(2\) as the points. Then find the value of the derivative at each point.

\[
f'(-2) = 3(-2)^2 - 3 = 9 \quad f'(0) = -3 \quad f'(2) = 3(2)^2 - 3 = 9
\]

Conclusion: \( f'(x) \) is positive when \( x \) is in the interval \((\infty, -1)\), negative when \( x \) is in \((-1, 1)\), and positive when \( x \) is in \((1, \infty)\). Therefore \( f(x) \) is increasing when \( x \) is in the interval \((\infty, -1)\), decreasing when \( x \) is in \((-1, 1)\), and increasing when \( x \) is in \((1, \infty)\).

(c) [4 pts] Find the relative maximum and relative minimum values of \( f(x) \).

Since \( f(x) \) increases as you approach \( x = -1 \) and then decreases afterwards, we can see that \( f(-1) = (-1)^3 - 3(-1) + 6 = 1 + 3 - 6 + 6 + 4 = 8 \) must be a relative maximum value.

Since \( f(x) \) decreases as you approach \( x = 1 \) and then increases afterwards, we can see that \( f(1) = (1)^3 - 3(1) + 6 = 1 - 3 + 6 + 4 = 8 \) must be a relative maximum value.

(d) [6 pts] Sketch the graph. There is a link to the graph on the class web page.

QUIZ 6 Solutions

1. (5 pts; p232 #10) \[
\lim_{{x \to -\infty}} \frac{4 - 3x - 12x^2}{1 + 5x + 3x^2} = \lim_{{x \to -\infty}} \frac{4x^2 - 3 - 12}{x^2 + 5 + 3} = \frac{0 - 0 - 12}{0 + 0 + 3} = \frac{-12}{3} = -4
\]

2. (20 pts; p215 #28) Let \( f(x) = \frac{-4}{x^2 + 1} \).

(a) [3 pts] Find all vertical and horizontal asymptotes.

To find vertical asymptotes, set the denominator equal to zero. Since \( x^2 + 1 = 0 \) has no solution, there are none. To find horizontal asymptotes, find \( \lim_{{x \to \infty}} f(x) \). You should get \( y = 0 \) as a horizontal asymptote.

(b) [4 pts] Find \( f'(x) \) and the critical points of \( f(x) \); find where \( f(x) \) is increasing and decreasing.

Using the quotient rule, \( f'(x) = \frac{(0)(x^2 + 1) - (-4)(2x)}{(x^2 + 1)^2} = \frac{8x}{(x^2 + 1)^2} \), and then setting \( f'(x) = 0 \) gives \( 8x = 0 \), since a fraction can only be equal to zero when its numerator is zero. The sign of \( f'(x) \) just depends on the sign of \( 8x \), since the denominator is always positive. This shows that \( f'(x) \) is negative when \( x < 0 \), and positive when \( x > 0 \), so \( f(x) \) is decreasing when \( x < 0 \), and increasing when \( x > 0 \).

(c) [4 pts] Find \( f''(x) \), and the intervals on which \( f(x) \) is concave up and concave down.

\[
f''(x) = \frac{(8)(x^2 + 1)^2 - (8x)(2)(x^2 + 1)(2x)}{((x^2 + 1)^2)^2} = \frac{8(x^2 + 1)(x^2 + 1) - (x)(2)(2x)}{(x^2 + 1)^4} = \frac{(8)(1 - 3x^2)}{(x^2 + 1)^3}
\]

Since the denominator is always positive, the second derivative can only change sign when the numerator is zero. Set the numerator equal to zero, then test the sign of \( f''(x) \) in the corresponding intervals.

\[
1 - 3x^2 = 0 \quad x^2 = \frac{1}{3} \quad x = \pm \sqrt{\frac{1}{3}} \quad \text{Intervals: } (-\infty, -\sqrt{\frac{1}{3}}), (-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}), (\sqrt{\frac{1}{3}}, \infty)
\]

\[
f''(-1) = \frac{(8)(1 - 3(-1)^2)}{((-1)^2 + 1)^3} = \frac{(8)(1 - 3)}{(1 + 1)^3} < 0 \quad f''(0) = \frac{(8)(1 - 0)}{(0 + 1)^3} > 0 \quad f''(1) = \frac{(8)(1 - 3)}{(1 + 1)^3} < 0
\]

Thus \( f(x) \) is concave down on \((-\infty, -\sqrt{\frac{1}{3}})\), concave up on \((-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})\), and concave down on \((\sqrt{\frac{1}{3}}, \infty)\).

(c) [4 pts] Find the relative maximum and relative minimum values of \( f(x) \).

The only critical point is \( x = 0 \). Since \( f(x) \) decreases till \( x = 0 \) and then increases, it must give a relative minimum. You can also see this from the fact that \( f(x) \) is concave up when \( x = 0 \). The minimum is \( f(0) = -4/1 = -4 \).

(d) [5 pts] There is a link to the graph on the class web page.