1. (20 pts) Let \( A \) be the following matrix. \( A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & -2 & -1 & 0 & -1 \\ -3 & 3 & 0 & -3 & -3 \end{bmatrix} \)

(a) Reduce the matrix \( A \) to row echelon form.
(b) Find a basis for the column space of \( A \).
(c) Find a basis for the nullspace of \( A \).

2. (20 pts) Let \( A \) be the following matrix. \( A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix} \)

(a) Find the characteristic polynomial of \( A \).
(b) Find the characteristic values of \( A \). \( \text{Hint:} \) One of the values is \( \lambda = 2 \).
(c) Why can the matrix \( A \) be diagonalized?

3. (25 pts) Let \( A \) be the following matrix. \( A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix} \)

You are given that \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \), \( \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) are eigenvectors of \( A \).

(a) Find the eigenvalues of \( A \) that correspond to \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), and \( \mathbf{v}_3 \).
(b) Find the inverse of the following matrix. \( P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \)

(c) For the matrix \( P \) in part (b), compute \( P^{-1}AP \). \( \text{Hint:} \) Your answer should be a diagonal matrix.

4. (25 pts) Let \( P_2 \) be the vector space of all polynomials of degree at most 2. Define the function \( L: P_2 \to P_2 \) by \( L(p(x)) = p(x) + x^2p'(x) \), for all polynomials \( p(x) \) in \( P_2 \).

(a) Show that \( L \) is a linear transformation.
(b) Find the matrix of \( L \) relative to the standard basis \( S = \{x^2, x, 1\} \).
(c) Find the rank and nullity of \( L \).

5. (25 pts) Let \( \mathbf{q}_1 = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}) \), \( \mathbf{q}_2 = (\frac{2}{7}, \frac{1}{7}, \frac{3}{7}) \), \( \mathbf{a}_3 = (1, 1, 1) \).

(a) Show that \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \) are orthogonal, and that each has length 1.
(b) Use the Gram–Schmidt process to transform the basis \( \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}_3\} \) into an orthonormal basis. \( \text{Hint:} \) By part (a) you only need to apply the Gram–Schmidt process to the third vector.
(c) Find the coordinates of \((1,0,1)\) relative to the orthonormal basis in part (b).

6. (25 pts) Let \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) be a basis for the vector space \( V \). Define \( \mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_3 \), \( \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2 \), \( \mathbf{u}_3 = \mathbf{v}_2 + \mathbf{v}_3 \).

(a) Show that \( \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \) is a basis for \( V \).
(b) Define a linear transformation \( L: V \to V \) by letting \( L(\mathbf{v}_1) = \mathbf{u}_1 \), \( L(\mathbf{v}_2) = \mathbf{u}_2 \), \( L(\mathbf{v}_3) = \mathbf{u}_3 \). Choose a basis \( S \) for \( V \) and find the matrix for \( L \) relative to \( S \).
(c) Find the rank and nullity of the linear transformation \( L \) defined in part (b).

7. (30 pts) For each of the following subsets, decide whether or not the subset is a subspace of the given vector space. If it \emph{is} a subspace, show that it satisfies the necessary conditions. If it \emph{is not} a subspace, explain why not.

(a) \( \{ (x, y, z) \mid 2x - 3y + 4z = 0 \} \) in \( \mathbb{R}^3 \).
(b) \( \{ p(x) \mid p(0) = 2 \} \) in the vector space \( P_2 \) of all polynomials of degree at most 2.
(c) The set of all diagonal \( 2 \times 2 \) matrices in the vector space \( M_{22} \) of all \( 2 \times 2 \) matrices.

8. (30 pts) For each of the following questions, write out a careful proof, in complete sentences.

(a) Let \( L_1: V_1 \to V_2 \) and \( L_2: V_2 \to V_3 \) be linear transformations. Prove that the composition \( L_2L_1 \), defined from \( V_1 \) to \( V_3 \), is also a linear transformation.
(b) Prove that if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal vectors, then \( ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \).
(c) Prove that if \( A \) and \( B \) are similar matrices, then \( \det(A) = \det(B) \).