1. (20 pts) Let $A$ be the following matrix. 

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & -2 & -1 & 0 & -1 \\ -3 & 3 & 0 & -3 & -3 \end{bmatrix}$$

(a) Reduce the matrix $A$ to row echelon form.

The leading 1’s occur in the 1st and 3rd columns, so these columns form a basis:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\
-2 \\ 0 \\ 1 \\ 0 \\
-3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

(b) Find a basis for the column space of $A$.

(c) Find a basis for the nullspace of $A$.

To diagonalize $A$ we need to be able to find 3 linearly independent eigenvectors, since $A$ is a $3 \times 3$ matrix. Theorems tell us this is possible because $A$ has 3 different eigenvalues.

2. (20 pts) Let $A$ be the following matrix. 

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix}$$

(a) Find the characteristic polynomial of $A$. We need to find $\det(\lambda I - A)$.

$$p(\lambda) = \lambda - 2 \quad -3 \quad 1 \\ 0 \quad \lambda \quad 4 \\ 0 \quad -1 \quad \lambda - 5 = (\lambda - 2)(\lambda^2 - 5\lambda + 4) = (\lambda - 2)(\lambda - 4)(\lambda - 1)$$

(b) Find the eigenvalues of $A$. The eigenvalues are the roots of $p(\lambda)$: $\lambda = 1, 2, 4$.

(c) Why can the matrix $A$ be diagonalized?

3. (25 pts) Let $A$ be the following matrix. 

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix}$$

You are given that $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are eigenvectors of $A$.

(a) Find the eigenvalues of $A$ that correspond to $v_1$, $v_2$, and $v_3$.

$$Av_1 = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2v_1 \quad \text{so the corresponding eigenvalue is } \lambda = 2.$$ 

$$Av_2 = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = 2v_2 \quad \text{so the corresponding eigenvalue is } \lambda = 2.$$ 

$$Av_3 = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -v_3 \quad \text{so the corresponding eigenvalue is } \lambda = -1.$$ 

(b) Find the inverse of the following matrix. 

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{Soln: } P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$
Here is the work showing how to compute the inverse:
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 2 & 0 & 1 & -2 \\
0 & 1 & 1 & 0 & 1 & -1
\end{bmatrix}
\]

(c) For the matrix \(P\) in part (b), compute \(P^{-1}AP\).
\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & -1 & -1 \\
0 & -3 & 5
\end{bmatrix}
\begin{bmatrix}
2 & -3 & 3 \\
0 & -4 & 6 \\
0 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & -1 & 1 \\
0 & -1 & 2 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
2 & 0 & -1 \\
0 & 2 & -2 \\
0 & 2 & -1
\end{bmatrix}
= \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

4. (25 pts) Let \(P_2\) be the vector space of all polynomials of degree at most 2. Define the function \(L: P_2 \to P_2\) by \(L(p(x)) = p(x) + x^2p''(x)\), for all polynomials \(p(x)\) in \(P_2\).

(a) Show that \(L\) is a linear transformation.

The following expressions are equal for all scalars \(a, b\) and all polynomials \(p(x), q(x)\) in \(P_2\).
\[
L(ap(x) + bq(x)) = (ap(x) + bq(x)) + x^2(ap''(x) + bq''(x)) = ap(x) + bq(x) + ax^2ap''(x) + bx^2q''(x)
\]
\[
aL(p(x)) + bL(q(x)) = a(p(x) + x^2p''(x)) + b(q(x) + x^2q''(x)) = ap(x) + ax^2p''(x) + bq(x) + bx^2q''(x)
\]

(b) Find the matrix of \(L\) relative to the standard basis \(S = \{x^2, x, 1\}\).
\[
L(x^2) = x^2 + x^2 \cdot 2 = 3 \cdot x^2 + 0 \cdot x + 0 \cdot 1 \\
L(x) = x + x^2 \cdot 0 = 0 \cdot x^2 + 1 \cdot x + 0 \cdot 1 \\
L(1) = 1 + x^2 \cdot 0 = 0 \cdot x^2 + 0 \cdot x + 1 \cdot 1
\]
The matrix of the transformation is \(A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\).

(c) Find the rank and nullity of the matrix you found in part (b).
The rank of \(A\) is 3, so its nullity is 0.

5. (25 pts) Let \(w_1 = (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3})\), \(w_2 = (\frac{2}{3}, \frac{1}{3}, \frac{3}{3})\), \(u_3 = (1, 1, 1)\).

(a) Show that \(w_1\) and \(w_2\) are orthogonal, and that each has length 1.
\[
w_1 \cdot w_1 = \left(\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}\right) \cdot \left(\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}\right) = \frac{1}{3} \cdot \frac{1}{3} + \frac{4}{3} \cdot \frac{4}{3} - \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{3} + \frac{16}{9} - \frac{4}{9} = 1, \text{ so } ||w_1|| = 1.
\]
\[
w_2 \cdot w_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{3}{3}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{3}{3}\right) = \frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{3}{3} \cdot \frac{3}{3} = \frac{4}{9} + \frac{1}{9} + 1 = 1, \text{ so } ||w_2|| = 1.
\]
\[
w_1 \cdot w_2 = \left(\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{3}{3}\right) = \frac{1}{3} \cdot \frac{2}{3} + \frac{4}{3} \cdot \frac{1}{3} - \frac{2}{3} \cdot \frac{3}{3} = \frac{2}{9} + \frac{4}{9} - 1 = 0, \text{ so } w_2 \text{ is orthogonal to } w_1.
\]

(b) Use the Gram–Schmidt process to transform the basis \(\{w_1, w_2, u_3\}\) into an orthonormal basis.
*Hint:* You only need to apply the Gram–Schmidt process to the third vector.
\[
v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 = (1, 1, 1) - ((1, 1, 1) \cdot (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3})) \cdot (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}) - ((1, 1, 1) \cdot (\frac{2}{3}, \frac{1}{3}, \frac{3}{3})) \cdot (\frac{2}{3}, \frac{1}{3}, \frac{3}{3})
\]
\[
= (1, 1, 1) - \frac{1}{3} \cdot (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}) - \frac{2}{3} \cdot (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}) = (1, 1, 1) - (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}) - (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})
\]
\[
w_3 = \frac{1}{\sqrt{3}}(-2, 1, 1) = (-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \quad (\text{Multiply } v_3 \text{ by } 9, \text{ then divide by its length}.)
\]

(c) Find the coordinates of \((1, 0, 1)\) relative to the orthonormal basis in part (b).

The coordinates \(c_1, c_2, c_3\) are
\[
c_1 = (1, 0, 1) \cdot w_1 = (1, 0, 1) \cdot (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}
\]
\[
c_2 = (1, 0, 1) \cdot w_2 = (1, 0, 1) \cdot (\frac{2}{3}, \frac{1}{3}, \frac{3}{3}) = \frac{2}{3} + \frac{1}{3} = \frac{4}{3}
\]
\[
c_3 = (1, 0, 1) \cdot w_3 = (1, 0, 1) \cdot (-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}
\]

6. (20 pts) For each of the following subsets, decide whether or not the subset is a subspace of the given vector space. If it is a subspace, show that it satisfies the necessary conditions. If it is not a subspace, explain why not.

(a) \(\{(x, y, z) \mid 2x - 3y + 4z = 0\}\) in \(\mathbb{R}^3\).

This is the solution space of a system of equations, so a theorem guarantees that it must be a subspace. To show this directly, suppose that \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) belong to the subset. Then...
2(x_1 + x_2) - 3(y_1 + y_2) + 4(z_1 + z_2) = (2x_1 - 3y_1 + 4z_1) + (2x_2 - 3y_2 + 4z_2) = 0 + 0 = 0, so the sum also belongs to the subset. For any scalar c, 2(cx_1) - 3(cy_1) + 4(cz_1) = c(2x_1 - 3y_1 + 4z_1) = c \cdot 0 = 0, and so (cx_1, cy_1, cz_1) belongs to the subset.

(b) \{p(x) \mid p(0) = 2\} in the vector space \( P_2 \) of all polynomials of degree at most 2.

This is not a subspace because the zero polynomial does not belong to it.

(c) The set of all diagonal \( 2 \times 2 \) matrices in the vector space \( M_{22} \) of all \( 2 \times 2 \) matrices.

The set is a subspace because \[
\begin{bmatrix}
    x & 0 \\
    0 & y
\end{bmatrix}
+ \begin{bmatrix}
    z & 0 \\
    0 & w
\end{bmatrix} = \begin{bmatrix}
    x + z & 0 \\
    0 & y + w
\end{bmatrix}
\]
and
\[
c \begin{bmatrix}
    x & 0 \\
    0 & y
\end{bmatrix} = \begin{bmatrix}
    cx & 0 \\
    0 & cy
\end{bmatrix}.
\]

7. \( 10 \) pts Define a linear transformation \( L : \mathbb{R}_3 \to \mathbb{R}_3 \) by letting \( L(v) \) be the projection of the vector \( v \) on the plane \( x + y + z = 0 \). Choose a basis \( S \) for \( \mathbb{R}_3 \), and find the matrix of \( L \) relative to the basis you choose.

\( \text{Hint: Consider choosing a basis that consists of two vectors in the plane, and one that is perpendicular to the plane.} \)

The equation is \( x = -y - z \), so \((-1, 1, 0)\) and \((-1, 0, 1)\) form a basis. The normal vector (perpendicular to the plane) is \((1, 1, 1)\). For this basis, \( L(-1, 1, 0) = (-1, 1, 0), L(-1, 0, 1) = (-1, 0, 1), \) and \( L(1, 1, 1) = (0, 0, 0) \).

The corresponding matrix is
\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]

8. \( 45 \) pts Answer 3 of the following 5 questions.

For each of the questions, write out a careful proof, in complete sentences.

(A) Prove that for any matrix \( A \), the matrix \((A^T)A\) is symmetric.

The matrix \( X \) is symmetric if \( X^T = X \). To check this for \( X = (A^T)A \), we have \((A^T)A = A^T(A^T)^T = A^T A \), which shows that \((A^T)A\) is symmetric.

(B) Prove that for any square matrix \( A \), the nullity of \( A^T \) is equal to the nullity of \( A \).

Assume that \( A \) is an \( n \times n \) matrix. The nullity of \( A \) is \( n \) minus the row rank of \( A \); the nullity of \( A^T \) is \( n \) minus the row rank of \( A^T \). But the row rank of \( A^T \) is the same as the column rank of \( A \), and this equals the row rank of \( A \). Therefore the nullity of \( A^T \) is equal to the nullity of \( A \).

(C) Let \( \{v_1, v_2, v_3\} \) be a basis for the vector space \( V \). Define \( u_1 = v_1 + v_3, u_2 = v_1 + v_2, u_3 = v_2 + v_3 \).

Show that \( \{u_1, u_2, u_3\} \) is a basis for \( V \).

Since we have 3 vectors in a 3-dimensional vector space, we only need to check that they are linearly independent, so we need to solve the equation \( c_1 u_1 + c_2 u_2 + c_3 u_3 = 0 \). Substituting gives
\[
c_1 (v_1 + v_3) + c_2 (v_1 + v_2) + c_3 (v_2 + v_3) = 0 \quad \text{or} \quad (c_1 + c_2)v_1 + (c_2 + c_3)v_2 + (c_1 + c_3)v_3 = 0.
\]

Now we get \( c_1 + c_2 = 0, c_2 + c_3 = 0, \) and \( c_1 + c_3 = 0 \) because the vectors \( v_1, v_2, v_3 \) are linearly independent. To solve the system, we can use the following matrix. Since it has rank 3, the only solution is \( c_1 = 0, c_2 = 0, c_3 = 0 \), and therefore the vectors \( u_1, u_2, u_3 \) are linearly independent.

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    1 & 0 & 1
\end{bmatrix} \sim> \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    0 & -1 & 1
\end{bmatrix} \sim> \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    0 & 0 & 2
\end{bmatrix}
\]

(D) Prove that if \( u \) and \( v \) are orthogonal vectors, then \( ||u + v||^2 = ||u||^2 + ||v||^2 \).

Assume that \( u \) and \( v \) are orthogonal vectors, so that \( \langle u, v \rangle = 0 \).

\[
||u + v||^2 = (u + v, u + v) = (u, u) + 2(u, v) + (v, v) = ||u||^2 + 0 + ||v||^2 = ||u||^2 + ||v||^2.
\]

(E) Prove that if \( A \) and \( B \) are similar matrices, then \( \det(A) = \det(B) \).

If \( A \) and \( B \) are similar matrices, then there is an invertible matrix \( P \) with \( A = P^{-1}BP \). Therefore
\[
\det(A) = \det(P^{-1}BP) = \det(P^{-1}) \det(B) \det(P) = \frac{1}{\det(P)} \det(B) \det(P) = \det(B).
\]

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