1. (15 pts) Define \( f : \mathbb{Z}_8 \to \mathbb{Z}_{12} \) by \( f([x]_8) = [3x]_{12} \), for all \([x]_8 \in \mathbb{Z}_8\).
   
   (a) Show that \( f \) is a well-defined function.
   
   \textit{Recall: you must show that if } x_1 \equiv x_2 \pmod{8}, \text{ then } 3x_1 \equiv 3x_2 \pmod{12}.
   
   (b) Find the image \( f(\mathbb{Z}_8) \) and the set of equivalence classes \( \mathbb{Z}_8/f \) defined by \( f \), and exhibit the one-to-one correspondence between these sets.

2. (25 pts) Let \( \sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9) \) and \( \tau = (1 \ 5 \ 4 \ 7 \ 2 \ 6 \ 8 \ 9 \ 3) \).
   
   (a) Write each of \( \sigma \), \( \tau \), \( \sigma \tau \), \( \tau \sigma \), and \( \sigma \tau \sigma^{-1} \) as a product of disjoint cycles.
   
   (b) Find the order of each of \( \sigma \), \( \tau \), \( \sigma \tau \), \( \tau \sigma \), and \( \sigma \tau \sigma^{-1} \).
   
   \textit{Recall: the order of a permutation } \sigma \textit{ is the smallest positive exponent } m \textit{ for which } \sigma^m \textit{ is equal to the identity.}
   
   (c) Determine whether each of \( \sigma \), \( \tau \), \( \sigma \tau \), \( \tau \sigma \), and \( \sigma \tau \sigma^{-1} \) is an even permutation or an odd permutation.

3. (20 pts) Let \( f : S \to T \) and \( g : T \to U \) be functions.
   
   (a) State these definitions: \( f \) is \textbf{one-to-one}; \( f \) is \textbf{onto}.
   
   (b) Prove that if \( gf \) is a one-to-one function, then so is \( f \).
   
   (c) Prove that if \( gf \) is an onto function, then so is \( g \).

4. (20 pts) For integers \( m, n, b \) with \( n > 1 \), define \( f : \mathbb{Z}_n \to \mathbb{Z}_n \) by \( f([x]_n) = [mx + b]_n \).
   
   You may assume that \( f \) is a well-defined function.
   
   Prove that \( f \) is a one-to-one correspondence if and only if \( \gcd(m,n) = 1 \). Then find the inverse function \( f^{-1} \), assuming that \( \gcd(m,n) = 1 \).

5. (10 pts) Let \( S \) be the set of all \( n \times n \) matrices with real entries. For \( A, B \in S \), define \( A \sim B \) if there exists an invertible matrix \( P \) such that \( B = PAP^{-1} \). Prove that \( \sim \) is an equivalence relation.

6. (10 pts) Let \( \sigma \in S_n \) have order \( m \). Prove that if \( k \) is any integer, then \( \sigma^k = (1) \) if and only if \( m \mid k \).
   
   \textit{The rules:} You must give a direct proof that does not use Proposition 2.3.7 from the text, which states that \( \sigma^i = \sigma^j \) if and only if \( i \equiv j \pmod{m} \).