1. (a) State the definition of a **group**.

   (b) State the definition of an **abelian** group.

   (c) Give an example of a finite group that is not abelian. Explain your answer.

2. (a) For each \( \sigma \in S_3 \), find \( \langle \sigma \rangle \), the cyclic subgroup generated by \( \sigma \).

   (b) Find the order of each element of the group \( \mathbb{Z}_4 \times \mathbb{Z}_4^\times \).

3. (a) What are the possibilities for the order of an element of \( \mathbb{Z}_{11}^\times \)? Explain your answer.

   (b) Show that \( \mathbb{Z}_{11}^\times \) is a cyclic group.

4. (a) In the group \( G = GL_2(\mathbb{R}) \) of invertible \( 2 \times 2 \) matrices with real entries, show that

   \[
   H = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(\mathbb{R}) \mid a_{11} = 1, a_{21} = 0, a_{22} = 1 \right\}
   \]

   is a subgroup of \( G \).

   (b) Show that \( H \) is isomorphic to the group \( \mathbb{R} \) of all real numbers, under addition.

5. **Choose Part A OR Part B.**

   **Part A.** Prove Proposition 3.4.5: Assume that \( m \) and \( n \) are positive integers such that \( \gcd(m, n) = 1 \). For \( k = mn \), define \( \phi : \mathbb{Z}_k \to \mathbb{Z}_m \times \mathbb{Z}_n \) by \( \phi([x]_k) = ([x]_m, [x]_n) \), for all \( [x]_k \in \mathbb{Z}_k \). Prove that \( \phi \) is a well-defined function, and that \( \phi \) is an isomorphism.

   **Part B.** State and prove Lagrange’s theorem.

   In your proof you may assume Lemma 3.2.9, which states that if \( H \) is a subgroup of a group \( G \), and for \( a, b \in G \) we define \( a \sim b \) if \( ab^{-1} \in H \), then \( \sim \) is an equivalence relation on \( G \).