1. (1.3, p 41, #5) Let $p, q$ be distinct prime numbers, and let $n = pq$. In $\mathbb{Z}_n^\times$, let

$$H = \{ [a] | a \equiv 1 \pmod{p} \} \quad \text{and} \quad K = \{ [b] | b \equiv 1 \pmod{q} \}.$$ 

Show that $HK = \mathbb{Z}_n^\times$.

2. (1.3, p 41, #8) This exercise concerns subgroups of $\mathbb{Z} \times \mathbb{Z}$.

(a) For each positive integer $n > 1$, let $C_n = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \pmod{n} \}$. Show that $C_n$ is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(b) Let $D = \{ (a, a) | a \in \mathbb{Z} \}$ be the “diagonal” subgroup of $\mathbb{Z} \times \mathbb{Z}$. Show that every subgroup of $\mathbb{Z} \times \mathbb{Z}$ that contains $D$ has the form $C_n$, for some positive integer $n$.

(c) Show that $C_n \cong \mathbb{Z} \times \mathbb{Z}$.

3. (1.3, p 42, #10) Without writing down all 60 elements of $A_5$, describe the possible cycle structures and how many of each kind there are.

4. (1.3, p 42, #20) Given a permutation $\sigma \in S_n$, state a criterion on the cycle structure of $\sigma$ that determines whether or not $\sigma = \tau^2$ for some $\tau \in S_n$. Use this criterion to find the values of $n$ for which the alternating group $A_n$ is precisely the set of squares in $S_n$.

5. (1.4, p 54, #5) Let $G$, $G_1$, and $G_2$ be groups.

(a) Show that if $\phi_1 : G \to G_1$ and $\phi_2 : G \to G_2$ are group homomorphisms, then so is $\phi : G \to G_1 \times G_2$ defined by $\phi(x) = (\phi_1(x), \phi_2(x))$, for all $x \in G$.

(b) Show that if $\phi : G \to G_1 \times G_2$ is any group homomorphism, then there exist group homomorphisms $\phi_1 : G \to G_1$ and $\phi_2 : G \to G_2$ such that $\phi$ has the form given in part (a).

6. (1.4, p 54, #13) Let $N$ be a normal subgroup of the group $G$. Prove that $G/N$ is abelian if and only if $N$ contains all elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

7. (1.4, p 55, #17) Show that $\text{SL}_2(\mathbb{Z}_3)$ is not isomorphic to $S_4$.

8. (1.4, p 54, #25) Let $G$ be a group for which the only isomorphism from $G$ into itself is the identity mapping. Prove that $x^2 = 1$ for all $x \in G$.

9. (1.4, p 54, #27) Let $G$ be a group, and let $N$ be a normal subgroup with $[G : N] = p$, where $p$ is prime. Show that if $H$ is a subgroup of $G$ that is not contained in $N$, then $HN = G$ and $[H : H \cap N] = p$.

10. (1.4, p 54, #29) Let $G$ be a group with subgroups $H$ and $K$.

(a) Show that if $[G : H]$ and $[G : K]$ are finite, then so is $[G : H \cap K]$.

(b) Show that if $H \subseteq K$ and both $[G : K]$ and $[K : H]$ are finite, then we have $[G : H] = [G : K][K : H]$.