4.3 #1: In Example 4.3.3 use a direct calculation to verify that the subfield fixed by \( \langle \alpha^3 \beta \rangle \) is \( \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2}) \).

4.3 #2: In Example 4.3.3 determine which subfields are conjugate, and in each case find an automorphism under which the subfields are conjugate.

4.4 #3: Find the Galois group of \( x^5 - 1 \) over \( \mathbb{Q} \).

4.4 #4: Find the Galois group of \( x^9 - 1 \) over \( \mathbb{Q} \).

4.4 #7: Let \( H \) be a subgroup of \( S_p \), where \( p \) is prime. Show that if \( H \) contains a transposition and a cycle of length \( p \), then \( H = S_p \).

4.4 #8: Prove that if \( f(x) \in \mathbb{Q}[x] \) is irreducible of prime degree \( p \) and has exactly two non-real roots in \( \mathbb{C} \), then the Galois group of \( f(x) \) over \( \mathbb{Q} \) is \( S_p \).

The remaining problems are worth 10 points each, instead of the usual 5 points.

IX #1. Let the field \( F \) be a finite, normal, separable extension of the field \( K \). Suppose that the Galois group of \( F \) over \( K \) is cyclic of order 50. Find how many distinct fields \( E \) there are with \( K \subseteq E \subseteq F \), and how many of these are normal extensions of \( K \).

IX #2. Find the Galois group of \( x^3 - 7 \) over \( \mathbb{Q} \).

IX #3. Let \( u = \sqrt{2} + \sqrt{2} \). Let \( f(x) \) be the minimal polynomial of \( \alpha \) over \( \mathbb{Q} \), and let \( F \) be the splitting field for \( f(x) \) over \( \mathbb{Q} \). Prove that \( \text{Gal}(F/\mathbb{Q}) \) is cyclic of order 4. Find all fields \( E \) with \( \mathbb{Q} \subseteq E \subseteq F \).

IX #4. Show that the Galois group of \( x^5 - 2 \) over \( \mathbb{Q} \) is \( F_{20} \). (This exercise is in Dummit and Foote.)

IX #5. This exercise (from Dummit and Foote) shows that \( \text{Gal}(\mathbb{R}/\mathbb{Q}) = \{1_{\mathbb{R}}\} \).

(a) Let \( \alpha \in \text{Gal}(\mathbb{R}, \mathbb{Q}) \). Show that \( \alpha \) maps squares to squares, and maps positive reals to positive reals. Conclude that \( a < b \) implies \( \alpha(a) < \alpha(b) \) for all \( a, b \in \mathbb{R} \).

(b) Prove that \( -\frac{1}{m} < a - b < \frac{1}{m} \) implies that \( -\frac{1}{m} < \alpha(a) - \alpha(b) < \frac{1}{m} \) for all \( a, b \in \mathbb{R} \). Conclude that \( \alpha \) is a continuous function on \( \mathbb{R} \).

(c) Prove that any continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) that is the identity on \( \mathbb{Q} \) is the identity mapping.