CHAPTER 1: RINGS
Review Problems

1. Let $I$ be an ideal of the commutative ring $R$. Prove that $I$ is a prime ideal if $I$ is the kernel of a ring homomorphism from $R$ into a field.

Solution: If $I = \ker(\phi)$, where $\phi : R \rightarrow F$ is a ring homomorphism, and $F$ is a field, then the fundamental homomorphism theorem implies that $R/I$ is isomorphic to a subring of the field $F$. It follows that $R/I$ is an integral domain, and hence $I$ is a prime ideal. (See Proposition 1.3.2 (b).)

Conversely, if $I$ is a prime ideal, then it is the kernel of the composite mapping defined by the projection $R \rightarrow R/I$ followed by the embedding of $R/I$ into its field of quotients $Q(R/I)$. (See Theorem 1.3.5 for the construction of the field of quotients.)

2. Let $R$ be a commutative ring that is not a field, and let $P$ be a maximal ideal of $R$. Let $I = P[x]$ be the ideal of the polynomial ring $R[x]$ consisting of all polynomials in $R[x]$ with coefficients in $P$. Show that $I$ is a prime ideal that is not maximal.

Solution: Let $F$ denote the factor ring $R/P$, which is a field. Define $\phi : R[x] \rightarrow F[x]$ by mapping $x$ to $x$ and mapping each coefficient $a$ to $a + P$. This is a well-defined ring homomorphism (see Example 1.2.1), and it is clear that it is onto, with kernel $I$, so $R[x]/I$ is isomorphic to $F[x]$. This shows that $I$ is a prime ideal, since $F[x]$ is an integral domain. On the other hand, $x$ generates a proper nonzero ideal in $F[x]$, so the corresponding preimage in $R[x]$ is an ideal that properly contains $I$. Thus $I$ is not a maximal ideal of $R[x]$.

Note: This implies that $R[x]$ is not a principal ideal domain, since in a principal ideal domain every nonzero prime ideal is maximal.

3. Let $D$ be a principal ideal domain, and let $P$ be a prime ideal of $D$. Prove that $D/P$ is a principal ideal domain.

Solution: Since $P$ is a prime ideal, the factor ring $D/P$ is an integral domain. By Proposition 1.2.9, each ideal of $D/P$ has the form $I/P$, for an ideal $I$ with $P \subseteq I \subseteq D$. By assumption $I = aD$ for some $a \in D$, so it follows that $I/P$ is a principal ideal since each coset $r + P$ has the form $ad + P$, for some $d \in D$.

4. Let $R$ be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of $R$ is prime, then $R$ is a field.

Solution: We first note that $R$ is an integral domain since the zero ideal is prime. Let $a$ be a nonzero element of $R$. By assumption, the ideal $a^2R$ is prime, and so $a^2 \in a^2R$ implies $a \in a^2R$. Thus $a = a^2r$ for some $r \in R$, and since $R$ is an integral domain, we can cancel $a$ to obtain $1 = ar$, showing that $a$ is invertible.

5. Let $F$ and $K$ be fields. Prove that if $F[x] \cong K[x]$, then $F \cong K$.

Solution: Let $\phi : F[x] \rightarrow K[x]$ be a ring isomorphism. First note that $\phi$ maps $0$ to $0$. The units of $F[x]$ are the nonzero elements of $F$. Since the corresponding statement holds in $K[x]$, and any isomorphism preserves units, it follows that $\phi$ maps $F$ to $K$. 
Thus $\phi$ restricts to a ring homomorphism from $F$ into $K$. Similarly, the inverse $\phi^{-1}$ restricts to a ring homomorphism from $K$ into $F$. It follows that the restriction of $\phi$ to $F$ is the required isomorphism from $F$ to $K$.

6. Let $F$ be a field. Show that in the factor ring $F[x]/(x^n)$ an element $f(x) + (x^n)$ is invertible iff $f(0) \neq 0$.

Solution: Note that the coset $x + (x^n)$ is a nilpotent element in the factor ring $F[x]/(x^n)$. Thus if $f(x) \in F[x]$ and $f(0) = 0$, then in the factor ring the element $f(x) + (x^n)$ is a sum of nilpotent elements. Exercise 1.4.1 of the text shows that in a commutative ring the set of nilpotent elements forms an ideal. If follows that if $f(0) = 0$, then $f(x) + (x^n)$ is a nilpotent element of $F[x]/(x^n)$, and therefore cannot be invertible.

Conversely, if $f(0) \neq 0$, then $f(x) + (x^n)$ is the sum of an invertible element and a nilpotent element. Therefore $f(x) + (x^n)$ is invertible in $F[x]/(x^n)$. (Exercise 1.1.9 in the text states that if $u$ is a unit and $a$ is nilpotent, then $u - a$ is a unit; the same statement holds for $u + a$.)

7. Let $F = \mathbb{Z}_2$ be the field with two elements, and let $R$ be the factor ring $F[x]/(x^2 + 1)$. Show that $R$ has four elements, but that it is not isomorphic (as a ring) to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Solution: The ring $R$ has four cosets, represented by $0$, $1$, $x$, and $x + 1$. Since $R$ has characteristic $2$, the underlying abelian group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $R$ cannot be isomorphic (as a ring) to $\mathbb{Z}_4$.

Since $F$ has characteristic $2$, $x^2 + 1 = (x + 1)^2$. Thus $x + 1$ is a nilpotent element, while $x$ is invertible, since $x^2 \equiv 1 \pmod{x^2 + 1}$. In $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ there are no nonzero nilpotent elements, and the only invertible element is $(1, 1)$, since $(1, 0)$ and $(0, 1)$ are zero divisors. Thus $R$ cannot be isomorphic to $\mathbb{Z}_4$.

8. Let $F = \mathbb{Z}_2$ be the field with two elements. Show that the ring $R = F[x]/(x^3 + x)$ has exactly four proper nontrivial ideals.

Solution: Over $F$, the polynomial $x^3 + x$ factors as $x(x + 1)^2$. The lattice of ideals of $R$ corresponds to the lattice of ideals of $F[x]$ that contain the ideal $I = (x^3 + x)$. Since $F[x]$ is a principal ideal domain, this lattice corresponds to the lattice of factors of $x^3 + x$. Thus the proper nontrivial ideals of $R$ are generated by the elements $x + 1$, $(x + 1)^2$, $x$, and $x(x + 1)$.

9. Show that if $R$ is a division ring, then for any $a \in R$ the centralizer $C(a)$ is a division ring.

Solution: An earlier exercise in the Section 1.1 of these class notes shows that the centralizer of an element $a$, defined by $C(a) = \{r \in R \mid ra = ar\}$, is a subring of $R$. We will repeat the proof. If $r, s \in C(a)$, then $ra = ar$ and $sa = as$. It follows that $(r+s)a = ra + sa = ar + as = a(r+s)$, $(rs)a = r(sa) = r(as) = (ra)s = (ar)s = a(rs)$, and $(-r)a = (-1 \cdot r)a = a(-1 \cdot r) = a(-r)$. Thus $C(x)$ is a subring of $R$. 


If \( y \in C(a) \) and \( y \neq 0 \), then \( y^{-1}a = y^{-1}a(yy^{-1}) = y^{-1}(ay)y^{-1} = y^{-1}(ad)y^{-1} = (y^{-1}a)dy^{-1} = ay^{-1} \), and so \( y^{-1} \in C(a) \), showing that \( C(a) \) is a division ring.

10. Let \( D \) be an integral domain for which \( IJ = I \cap J \) for all ideals \( I, J \) of \( D \). Prove that \( D \) is a field.

Solution: Let \( a \in D \). Then \((aD)(aD) = aD \cap aD = aD\), so \( a \in (aD)(aD) \), which implies that \( a = \sum_{i=1}^{n} (ab_i)(ad_i) \) for some \( b_i, d_i \in D \), for \( 1 \leq i \leq n \). Thus \( a = a^2r \) for \( r = (\sum_{i=1}^{n} b_id_i) \). If \( a \neq 0 \), then we can cancel \( a \) to obtain \( 1 = ar \), showing that \( a \) is invertible.