Theoretical Assignment 5

Chapter 5

5.1

(a) Show that an elementary lower triangular matrix of type $k$ defined by (5.1) has the form

$$M_k = I + me_k^T$$

where $m = (0, 0, \ldots, 0, m_{k+1,k}, \ldots, m_{n,k})^T$.

(b) Show that the inverse of $M_k$ in (a) is given by

$$M_k^{-1} = I - me_k^T.$$ 

(c) Show that the elementary matrix $M$ defined by (5.2) is such that $Ma$, where $a = (a_{11}, a_{21}, \ldots, a_{n1})^T$, is a multiple of $e_1$.

5.2

(a) Given

$$
\begin{pmatrix}
1 \\
0.00001
\end{pmatrix}
$$

Using three-digit arithmetic, find an elementary matrix $M$ such that $Ma$ is a multiple of $e_1$.

(b) Using your computation in (a), find the $LU$ factorization of

$$A = \begin{pmatrix}
0.00001 & 1 \\
1 & 2
\end{pmatrix}.$$ 

(c) Let $\hat{L}$ and $\hat{U}$ be the computed $L$ and $U$ in part (b). Find

$$
(i) \quad \frac{\|A - \hat{L}\hat{U}\|_F}{\|A\|_F}, \quad (ii) \quad \frac{\|\hat{L}\|_F \|\hat{U}\|_F}{\|A\|_F}
$$

5.4

Assuming that $LU$ factorization of $A$ exists, prove that

(a) ($LDU$ factorization) $A$ can be written in the form

$$A = LDU_1$$

where $D$ is diagonal and $L$ and $U_1$ are unit lower and upper triangular matrices, respectively.

(b) ($LDL^T$ factorization) If $A$ is symmetric, the

$$A = LDL^T.$$
(c) Using (b), prove that if $A$ is symmetric and positive definite, then

$$A = HH^T$$

where $H$ is a lower triangular matrix with positive diagonal entries. This is known as the Cholesky decomposition.

### 5.11

Prove that the matrix $L$ in the factorization $PA = LU$, obtained by using Gaussian elimination with partial and complete pivoting, respectively, is unit lower triangular.

(5.13)

(a) Find a permutation matrix $P$, a unit lower triangular matrix $L$ and an upper triangular matrix $U$ such that $PA = LU$ for each of the following matrices.

(i) $A = \begin{pmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 1 & 4 \\ 3 & 1 & 4 & 1 & 5 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 100 & 99 & 98 \\ 98 & 55 & 11 \\ 0 & 1 & 1 \end{pmatrix}$

(v) $A$ of the form (5.7) with $n = 5$.

(c) Compute the growth factor in each case and verify the results on upper bounds of the growth factor in each case given in Section 5.3.

(d) Estimate the backward error for each of the factorizations.

### 5.15

(a) Prove that the growth factor $\rho \leq 2^{n-1}$ for GEPP applied to an $n \times n$ matrix.

(b) Construct a small example to show that for GE without pivoting the ratio $\frac{\|L\|\|U\|}{\|A\|}$ can be arbitrarily large.